

Survival probabilities of autoregressive processes

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Abstract

Given an autoregressive process X of order p (i.e. $X_n = a_1X_{n-1} + \dots + a_pX_{n-p} + Y_n$ where the random variables Y_1, Y_2, \dots are i.i.d.), we study the asymptotic behaviour of the probability that the process does not exceed a constant barrier up to time N (survival or persistence probability). Depending on the coefficients a_1, \dots, a_p and the distribution of Y_1 , we state conditions under which the survival probability decays polynomially, faster than polynomially or converges to a positive constant. Special emphasis is put on AR(2) processes.

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1 Introduction

For fixed $p \geq 1$, define $X_n = \sum_{k=1}^p a_k X_{n-k} + Y_n$, $n \geq 0$ with the convention that $X_n = 0$ for $n \leq 0$. Throughout the paper, we assume that $(Y_n)_{n \geq 1}$ is a sequence of i.i.d. (nondegenerate) random variables. $(X_n)_{n \geq 1}$ is called an autoregressive process of order p (AR(p)-process). We sometimes refer to the random variables $(Y_n)_{n \geq 1}$ as innovations. Denote by $p_N(x)$ the probability that the process X stays below x until time N , i.e.

$$p_N(x) := \mathbb{P} \left(\sup_{n=1, \dots, N} X_n \leq x \right), \quad N \geq 1, x \geq 0.$$

We refer to p_N as the survival probability up to time N , and we write p_N instead of $p_N(0)$ in the sequel.

The aim of this paper is to study the asymptotic behaviour of $p_N(x)$ as $N \rightarrow \infty$. Sometimes, the problem of determining the asymptotic behaviour of $p_N(x)$ is referred to as one-sided exit or one-sided barrier problem since $p_N(x) = \mathbb{P}(\tau_x > N)$ where $\tau_x := \inf \{n \geq 0 : X_n > x\}$. Such asymptotic results are known in a number of special cases such as random walks, integrated random walks, fractional Brownian motion and AR(1)-processes. The study of survival probabilities is motivated by several applications such as the inviscid Burgers equation (Sinaï (1992)) or zeros of random polynomials (Dembo et al. (2002)). We refer to the recent survey of Aurzada and Simon (2012), Li and Shao (2004) and Li and Shao (2005) for further information, applications and references. For instance, if X is a random walk ($p = 1, a_1 = 1$), it holds that $p_N(x) \sim c(x)N^{-1/2}$ if $\mathbb{E}[Y_1] = 0$ and $\mathbb{E}[Y_1^2] = 1$ (see e.g. Feller (1971)). Novikov and Kordzakhia (2008) study AR(1)-processes with $a_1 \in (0, 1)$ and show that $p_N(x)$ decays at least exponentially for a large class of distributions. Bounds on the exponential rate of decay for AR(1)-processes with Gaussian innovations can be found in Aurzada and Baumgarten (2011). Besides, the decay of the survival probability is known for integrated random walks ($p = 2, a_1 = 2, a_2 = -1$): if $\mathbb{E}[Y_1] = 0$, $\mathbb{E}[Y_1^2] \in (0, \infty)$, it holds that $p_N(x) \asymp N^{-1/4}$ (see Dembo et al. (2012) and the references therein).

Taken as a whole, very little is known about the decay of p_N for AR processes except in the few cases mentioned above. As noted in Dembo et al. (2012), this would be of much interest in view of the frequent appearance of AR-processes and survival probabilities in physical and economic models. Here we investigate the behaviour of the survival probability for such processes under various conditions on the distribution of the innovations. Since an AR(p)-process X can be written as $X_n = \sum_{k=1}^n c_{n-k} Y_k$ where the (c_n) solve the difference equation $c_n = a_1 c_{n-1} + \dots + a_p c_{n-p}$ with suitable initial conditions, we search criteria for the sequence (c_n) that allow us to characterize the survival probability. Specifically, we are interested in the following question for AR(p)-processes: when is p_N of polynomial order, when does p_N converge to a positive limit and when is the decay faster than any polynomial? This classification seems natural if one recalls the results for AR(1)-processes $X_n = \rho X_{n-1} + Y_n$ where $c_n = \rho^n$ for all n . In this case, the behaviour of the survival probability ranges from exponential decay for

$\rho < 1$, polynomial decay if $\rho = 1$ and $\mathbb{E}[Y_1] = 0$ to convergence to a positive constant if $\rho > 1$.

As we will see, the sequence (c_n) often has a much more complex form if $p \geq 2$ so that results for AR(1)-processes generally cannot be extended directly to higher order processes. We will derive criteria that allow for the classification of the asymptotic behaviour of the p_N as above. Particular emphasis is put on AR(2)-processes.

Let us introduce some notation and conventions: If $f, g : \mathbb{R} \rightarrow \mathbb{R}$ are two functions, we write $f \lesssim g$ ($x \rightarrow \infty$) if $\limsup_{x \rightarrow \infty} f(x)/g(x) < \infty$ and $f \asymp g$ if $f \lesssim g$ and $g \lesssim f$. Moreover, $f \sim g$ ($x \rightarrow \infty$) if $f(x)/g(x) \rightarrow 1$ as $x \rightarrow \infty$. If $(X_t)_{t \geq 0}$ is a stochastic process, it will often be convenient to write $X(t)$ instead of X_t . If X and Y are random variables, we write $X \stackrel{d}{=} Y$ to denote equality in distribution.

The remainder of this article is organized as follows. After presenting the main results for AR(2) processes below, we start some preliminaries on autoregressive processes in Section 2. In Section 3, we state general conditions ensuring that p_N decays exponentially or at least faster than any polynomial. Special emphasis is put on the case that $(c_n)_{n \geq 0}$ is absolutely summable and AR(2)-processes. We also prove exponential lower bounds for certain classes of AR-processes. We then determine the region where the survival probability decays polynomially for AR(2)-processes in Section 4, before briefly treating the case that p_N converges to a positive constant in Section 5.

1.1 Main results for AR(2) processes

Let us illustrate our main result when X is AR(2), i.e. $X_n = a_1 X_{n-1} + a_2 X_{n-2} + Y_n$ with $(Y_n)_{n \geq 1}$ i.i.d. Recall that $X_n = \sum_{k=1}^n c_{n-k} Y_k$ for $n \geq 1$. We decompose \mathbb{R}^2 into three disjoint regions C, E and P (see Figure 1.1) defined as follows:

$$\begin{aligned} C &:= \{(a_1, a_2) : a_1 \geq 2, a_1^2 + 4a_2 > 0\} \cup \{(a_1, a_2) : a_1 \in (0, 2), a_1 + a_2 > 1\} \\ &\quad \cup \{(a_1, a_2) : a_1^2 + 4a_2 = 0, a_1 > 2\} \cup \{(a_1, a_2) : a_1 = 0, a_2 > 1\}, \\ P &:= \{(a_1, a_2) : a_1 + a_2 = 1, a_2 \in [-1, 1]\}, \\ E &:= \mathbb{R}^2 \setminus (C \cup P). \end{aligned}$$

Depending on the membership of (a_1, a_2) to one of these sets, we can characterize the behaviour of the survival probability under certain conditions on the law of Y_1 .

If $(a_1, a_2) \in P$, the survival probability decays polynomially if $\mathbb{E}[Y_1] = 0$ under suitable moment conditions. The choice $a_1 = 2, a_2 = -1$ corresponds to an integrated random walk where $p_N \asymp N^{-1/4}$ if $\mathbb{E}[Y_1] = 0$ and $\mathbb{E}[Y_1^2] \in (0, \infty)$, see Dembo et al. (2012). If $a_1 + a_2 = 1$ with $|a_2| < 1$, we will see that X can be seen as a perturbed random walk since $c_n = c + C\epsilon^n$ where $|\epsilon| < 1$. Moreover, X can also be written as an integrated AR(1)-process. The process corresponding to $a_1 = 0, a_2 = 1$ describes two independent random walks such that its survival probability is the square of that of a random walk.

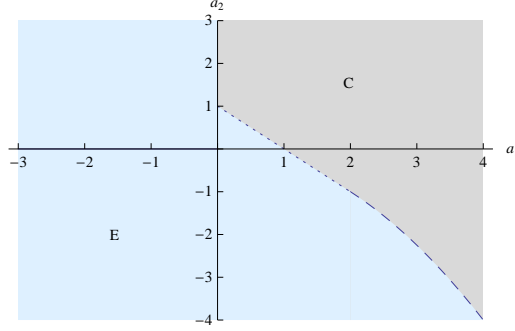


Figure 1: The regions C and E . P corresponds to the dotted line. The dashed line is the boundary of C whereas E is open.

Theorem 1.1. *Let $(a_1, a_2) \in P \setminus \{(2, -1)\}$. Assume that $\mathbb{E}[Y_1] = 0$ and that $\mathbb{E}[e^{|Y_1|^\alpha}] < \infty$ for some $\alpha > 0$. Then*

$$p_N = N^{-1/2+o(1)} \quad (|a_2| < 1), \quad p_N \asymp N^{-1} \quad (a_2 = 1).$$

Next, we also prove that the survival probability decays faster than any polynomial if $(a_1, a_2) \in E$ under certain conditions on the law of Y_1 .

Theorem 1.2. *Let $(a_1, a_2) \in E$. Assume that $\mathbb{P}(Y_1 > 0) \in (0, 1)$, $\mathbb{E}[e^{|Y_1|^\alpha}] < \infty$ for some $\alpha > 0$ and that the characteristic function φ of Y_1 satisfies $\varphi(t) \rightarrow 0$ as $|t| \rightarrow \infty$. Then $p_N \lesssim \exp(-\lambda N / \log N)$ for some $\lambda = \lambda(a_1, a_2) > 0$.*

Actually, we can show that p_N decays at least exponentially on most parts of E under various conditions on Y_1 . The reason for the rapid decay of the survival probability on E can be explained as follows: either $c_n \rightarrow 0$ exponentially fast or (c_n) oscillates and diverges to $\pm\infty$.

If $(a_1, a_2) \in C$, we will see that $c_n = \exp(\lambda n(1 + o(1)))$ for some $\lambda > 0$. One therefore expects that the process stays below a constant barrier at all times with positive probability. This is confirmed by the following theorem:

Theorem 1.3. *Let $(a_1, a_2) \in C$. Assume that $\mathbb{P}(Y_1 < 0) > 0$ and $\mathbb{P}(Y_1 \geq x) \lesssim (\log x)^{-\alpha}$ as $x \rightarrow \infty$ for some $\alpha > 1$. Then it holds that*

$$\mathbb{P}\left(\sup_{n \geq 1} X_n \leq x\right) = \lim_{N \rightarrow \infty} p_N(x) > 0, \quad x \geq 0.$$

Note that the assumption $\mathbb{E}[Y_1] = 0$ is essential for the polynomial behaviour of p_N if $(a_1, a_2) \in P$. For instance, if $(S_n)_{n \geq 1}$ is a random walk, it is known that the survival probability can decay polynomially or exponentially if $\mathbb{E}[S_1] > 0$ (see Doney (1989)) whereas it converges to a positive constant if $\mathbb{E}[S_1] < 0$. In contrast, if $(a_1, a_2) \in E \cup C$, the behaviour of p_N is more stable in the sense that Theorem 1.3 and Theorem 1.2 do

not rely on the condition $\mathbb{E}[Y_1] = 0$.

The best results can be obtained if the innovations are Gaussian, where we can actually prove that p_N admits an exponential upper bound for all $(a_1, a_2) \in E$. Summing up, this leads to the following theorem:

Theorem 1.4. *If Y_1 is Gaussian with zero mean, the following statements hold:*

1. $\lim_{N \rightarrow \infty} p_N = p_\infty > 0$ if and only if $(a_1, a_2) \in C$,
2. $p_N \sim cN^{-1}$ iff $(a_1, a_2) = (0, 1)$ and $p_N \asymp N^{-1/4}$ iff $(a_1, a_2) = (2, -1)$,
3. $p_N = N^{-1/2+o(1)}$ if and only if $(a_1, a_2) \in P$ and $|a_2| < 1$, and
4. $p_N \lesssim e^{-\lambda N}$ for some $\lambda > 0$ if and only if $(a_1, a_2) \in E$.

The theorems above are mostly corollaries to more general theorems that are also applicable to AR(p)-processes if $p \geq 3$ (see e.g. Theorem 3.2 and 3.10 and Proposition 3.17 and 5.1 below). We will indicate possible extensions throughout the article. The main advantage of focussing on AR(2)-processes consists of the fact that we have an explicit solution of the difference equation for the sequence $(c_n)_{n \geq 0}$. For instance, this allows us to explicitly describe the parameters (a_1, a_2) such that $c_n \rightarrow 0$.

Even for AR(2)-processes, one is forced to distinguish a variety of cases that require different treatment. It is clear that this becomes much more complicated for processes of higher order. Finally, let us mention that the class of AR(p)-processes contains p -times integrated centered random walks $S^{(p)}$ as a special case (i.e. $S^{(1)}$ is a centered random walk, and $S_n^{(p)} = \sum_{k=1}^n S_k^{(p-1)}$). Here, the behaviour of the survival probability is not known for $p \geq 3$.

2 Autoregressive processes

We begin by recalling a few facts about AR(p)-processes. For fixed $p \geq 1$, define $X_n = \sum_{k=1}^p a_k X_{n-k} + Y_n$, $n \geq 0$ with the convention that $X_n = 0$ for $n \leq 0$ where $(Y_n)_{n \geq 1}$ is a sequence of i.i.d. random variables. One verifies that $X_n = \sum_{k=1}^n c_{n-k} Y_k$ where

$$c_n = 0, \quad n < 0, \quad c_0 = 1, \quad c_n = \sum_{k=1}^p a_k c_{n-k}, \quad n \geq 1.$$

In other words, $(c_n)_{n \geq 0}$ solves the linear difference equation

$$c_n = a_1 c_{n-1} + \dots + a_p c_{n-p}, \quad n \geq p,$$

with initial conditions

$$c_0 = 1, \quad c_1 = a_1 c_0, \quad c_2 = a_1 c_1 + a_2 c_0, \quad \dots, \quad c_{p-1} = a_1 c_{p-2} + \dots + a_{p-1} c_0.$$

Solving this equation amounts to finding the roots $s_1, \dots, s_p \in \mathbb{C}$ of the characteristic polynomial $f_p(\cdot)$, given by $f_p(x) := x^p - \sum_{k=1}^p a_k x^{p-k}$, $x \in \mathbb{R}$. If $p = 2$, the roots s_1, s_2 of $f_2(\lambda) = \lambda^2 - a_1\lambda - a_2$ are given by

$$s_1 := (a_1 + h)/2, \quad s_2 := (a_1 - h)/2, \quad h := \sqrt{a_1^2 + 4a_2} \in \mathbb{C}. \quad (1)$$

Taking into account the initial conditions $c_0 = 1, c_1 = a_1$, one can show that

$$c_n = \begin{cases} h^{-1} (s_1^{n+1} - s_2^{n+1}), & n \geq 0, \quad a_1^2 + 4a_2 \neq 0, \\ (a_1/2)^n (n+1), & n \geq 0, \quad a_1^2 + 4a_2 = 0. \end{cases} \quad (2)$$

If $a_1^2 + 4a_2 < 0$, writing $s_1 = re^{i\varphi}$ and $s_2 = re^{-i\varphi}$ in polar form, elementary manipulations show that the solution is given by

$$c_n = |a_2|^{n/2} \left(\cos(n\varphi) + \frac{a_1}{\tilde{h}} \sin(n\varphi) \right), \quad (3)$$

where

$$\tilde{h} = \sqrt{-(a_1^2 + 4a_2)}, \quad \varphi = \begin{cases} \arctan(\tilde{h}/a_1) \in (0, \pi/2), & a_1 > 0, \\ \pi/2, & a_1 = 0, \\ \pi + \arctan(\tilde{h}/a_1) \in (\pi/2, \pi), & a_1 < 0. \end{cases}$$

Remark 2.1. It holds that $c_n \rightarrow 0$ if and only if $\max\{|s_1|, |s_2|\} < 1$ which is equivalent to the conditions

$$a_1 + a_2 < 1, \quad a_2 < 1 + a_1, \quad a_2 > -1,$$

see Theorem 2.37 of Elaydi (1999).

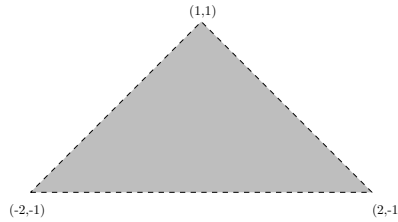


Figure 2: The region of parameters (a_1, a_2) where $c_n \rightarrow 0$

Remark 2.2. Note that the convention that $X_n = 0$ for $n < 0$ is not standard to define autoregressive processes. It is often customary to define AR(p)-processes as follows, see e.g. Chapter 3 in Brockwell and Davis (1987): If $(Y_n)_{n \in \mathbb{Z}}$ is a sequence of i.i.d. random variables, $X = (X_n)_{n \in \mathbb{Z}}$ is AR(p) if

$$X_n = a_1 X_{n-1} + \dots + a_p X_{n-p} + Y_n, \quad n \in \mathbb{Z}.$$

Moreover, X is called causal if there exists a deterministic sequence $(c_n)_{n \geq 0}$ with $\sum |c_n| < \infty$ such that $X_n = \sum_{k=0}^{\infty} c_k Y_{n-k}$.

By Theorem 3.1.1 of Brockwell and Davis (1987), X is causal if and only if the polynomial $p(z) = 1 - a_1 z - \dots - a_p z^p$ has no zeros in $\{z \in \mathbb{C} : |z| \leq 1\}$. In that case, the coefficients c_n are determined by the following relation $\sum_{k=0}^{\infty} c_k z^k = 1/p(z)$ for $|z| \leq 1$. Equating the coefficients of z^k , one easily verifies (or see Section 3.3 in Brockwell and Davis (1987)) that the sequence $(c_n)_{n \geq 0}$ satisfies the same recursion equation with the same initial conditions as above. Hence, if X is a causal AR(p)-process, we can decompose it for $n \geq 1$ in the following way:

$$X_n = \sum_{k=0}^{n-1} c_k Y_{n-k} + \sum_{k=n}^{\infty} c_k Y_{n-k} = \sum_{k=1}^n c_{n-k} Y_k + \sum_{k=0}^{\infty} c_{n+k} Y_{-k} = X_n^{(1)} + X_n^{(2)}.$$

Note that $X^{(1)}$ and $X^{(2)}$ are independent and that $X^{(1)}$ is an AR(p)-process in the sense of this article. The term $X^{(2)}$ can be seen as a small perturbation since $\mathbb{E} \left[\left| X_n^{(2)} \right| \right] \leq \mathbb{E} [|Y_1|] \sum_{k=n}^{\infty} c_k \rightarrow 0$.

Moreover, the fact that $c_n \rightarrow 0$ allows us to apply Theorem 3.5 below if X is AR(p) in the sense of Brockwell and Davis. By using the alternative definition above, we can also define autoregressive processes when c_n does not go to zero and we get a much larger class of processes including, for example, random walks.

We will use different methods to prove certain statements about the survival probability depending on the parameters (a_1, a_2) . To this end, set

$$\begin{aligned} E_1 &:= \{(a_1, a_2) : a_1 < 0, a_2 > 0, a_2 > 1 + a_1\}, \quad E_2 := (-\infty, 0]^2, \\ E_3 &:= \{(a_1, a_2) : a_1 > 0, a_1^2 + 4a_2 < 0\}. \end{aligned}$$

Figure 2 will be helpful to visualize the regions that will be considered separately below.

Let us also comment briefly on the dependence of the survival probability on the barrier x for AR(p)-processes. In principle, the behaviour of the survival probability can vary significantly for different barriers. An extreme example is an AR(1)-process $Z_n = \rho Z_{n-1} + Y_n$ where $\rho \in (0, 1)$ with $\mathbb{P}(Y_1 = 1) = \mathbb{P}(Y_1 = -1) = 1/2$. It is known that $p_N \lesssim \exp(-\lambda N)$ for some $\lambda > 0$ (see Theorem 3.1 below), whereas $p_N(x) = 1$ for all $x \geq 1/(1 - \rho)$ since $|X_n| = \left| \sum_{k=1}^n \rho^{n-k} Y_k \right| \leq \sum_{k=0}^{\infty} \rho^k = 1/(1 - \rho)$.

On the other hand, if $c_n \geq \delta > 0$ for all $n \geq 0$ and $\mathbb{P}(Y_1 \leq -\epsilon) > 0$ for some $\epsilon > 0$, one can show that $p_N(x) \asymp p_N$ as $N \rightarrow \infty$ for all $x \geq 0$. Indeed, note that if $Y_1 \leq -\epsilon$, it follows that $X_n = c_{n-1} Y_1 + \sum_{k=2}^n c_{n-k} Y_k \leq -\epsilon \delta + \sum_{k=2}^n c_{n-k} Y_k$, so that

$$\begin{aligned} p_N &= \mathbb{P} \left(\sup_{n=1, \dots, N} X_n \leq 0 \right) \geq \mathbb{P}(Y_1 \leq -\epsilon) \mathbb{P} \left(\sup_{n=2, \dots, N} \sum_{k=2}^n c_{n-k} Y_k \leq \delta \epsilon \right) \\ &\geq \mathbb{P}(Y_1 \leq -\epsilon) p_N(\delta \epsilon). \end{aligned}$$

Iteration shows that $p_N \geq \mathbb{P}(Y_1 \leq -\epsilon)^L p_N(L\delta\epsilon)$ for $L = 1, \dots, N$. Hence, if $x \geq 0$, take L with $L\delta\epsilon \geq x$ to get that $\mathbb{P}(Y_1 \leq -\epsilon)^L p_N(x) \leq p_N \leq p_N(x)$ for all N large enough.

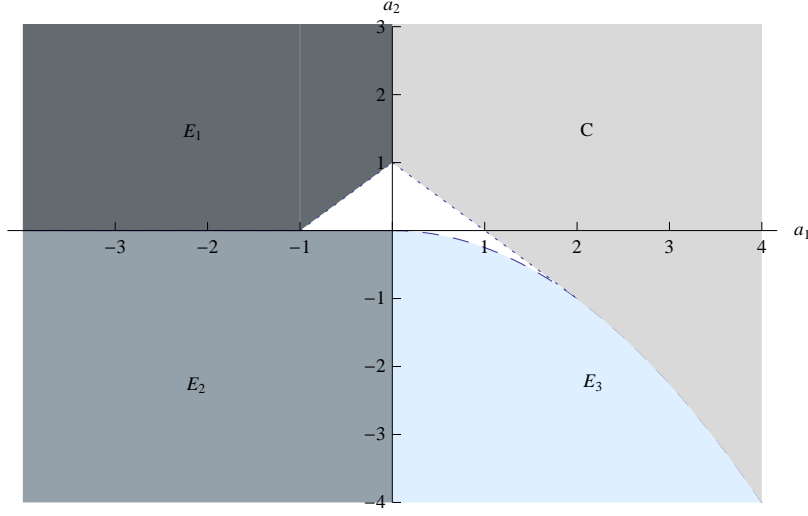


Figure 3: The regions E_1, E_2, E_3 and C

3 Exponential bounds

3.1 Exponential upper bounds

Let us begin with a trivial observation: If $a_1 \leq 0, \dots, a_p \leq 0$, we have that

$$\mathbb{P} \left(\sup_{n=1, \dots, N} X_n \leq 0 \right) \leq \mathbb{P} (Y_1 \leq 0)^N,$$

since $X_1 \leq 0, \dots, X_n \leq 0$ implies that $Y_k \leq -a_1 X_{k-1} - \dots - a_p X_{k-p} \leq 0$ for all $k = 1, \dots, n$. If $p = 2$, this shows that p_N decays exponentially on E_2 , see Figure 2.

As we will see in the sequel, exponential decay of p_N occurs for two different reasons: first, if $c_n \rightarrow 0$ and second, if $(c_n)_{n \geq 0}$ oscillates and diverges exponentially fast.

Let us first consider the case that c_n goes to zero. Recall that for AR(1)-processes $(Z_n)_{n \geq 1}$ with $Z_n = \rho Z_{n-1} + Y_n$ for $\rho \in (0, 1)$, $c_n = \rho^n \rightarrow 0$ and p_N decays exponentially under mild assumptions on the distribution of Y_1 by Theorem 1 of Novikov and Kordzakhia (2008):

Theorem 3.1. *Let $0 < \rho < 1$, $x \geq 0$ and assume that $\mathbb{E}[(Y_1^-)^\delta] < \infty$ for some $\delta \in (0, 1)$ and $\mathbb{P}(Y_1 > x(1 - \rho)) > 0$. Then $\mathbb{E}[\exp(\alpha \tau_x)] < \infty$ for some $\alpha > 0$.*

We now state a similar weaker result that provides a simple criterion for AR(p)-processes to ensure that p_N decays faster to zero than any polynomial.

Theorem 3.2. *Let $(c_k)_{k \geq 0}$ denote a sequence with $c_0 = 1$ and $A := \sum_{k=0}^{\infty} |c_k| < \infty$ such that $\sum_{k=q}^{\infty} |c_k| \leq C e^{-\lambda q}$ for every $q \geq 1$ ($C, \lambda > 0$ constants). Assume that there is $\delta > 0$ with $\mathbb{P}(Y_1 < -\delta) > 0$ and $\mathbb{P}(Y_1 > \delta) > 0$ and that $\mathbb{E}[Y_1^2] < \infty$. Let $X_n = \sum_{k=1}^n c_{n-k} Y_k$.*

Then for $x \in [0, \delta A)$, there is $c(x) > 0$ such that

$$p_N(x) \lesssim \exp\left(-c(x) \sqrt{N}\right), \quad N \rightarrow \infty.$$

Moreover, if $\mathbb{E}[\exp(|Y_1|^\alpha)] < \infty$ ($\alpha > 0$) and $x \in [0, \delta A)$, there is $c(x) > 0$ such that

$$p_N(x) \lesssim \exp(-c(x) N / \log N), \quad N \rightarrow \infty.$$

Proof. For $q \geq 1$, define $Z_{q,n} = \sum_{k=n-q}^n c_{n-k} Y_k$ for $n \geq q+1$. Note that $Z_{q,n}$ is measurable w.r.t. $\sigma(Y_{n-q}, \dots, Y_n)$ which implies that $(Z_{q,n(q+1)})_{n \geq 1}$ defines a sequence of i.i.d. random variables with $Z_{q,q+1} = X_{q+1}$. We will show that $Z_{q,n}$ is a good approximation of X_n if q is large. We then obtain an estimate on $p_N(x)$ by computing the survival probability of the independent r.v. $(Z_{q,(q+1)n})_{n \geq 1}$.

First, observe that

$$\begin{aligned} \mathbb{P}\left(\sup_{n=q+2, \dots, N} |X_n - Z_{q,n}| > u\right) &\leq \sum_{n=q+2}^N \mathbb{P}\left(\left|\sum_{k=1}^{n-q-1} c_{n-k} Y_k\right| > u\right) \\ &= \sum_{n=q+2}^N \mathbb{P}\left(\left|\sum_{k=q+1}^{n-1} c_k Y_k\right| > u\right) =: h_N(u). \end{aligned} \quad (4)$$

In the first equality, we have used that the Y_k are i.i.d., and therefore exchangeable. Hence,

$$\begin{aligned} \mathbb{P}\left(\sup_{n=1, \dots, N} X_n \leq x\right) &\leq \mathbb{P}\left(\sup_{n=q+2, \dots, N} Z_{q,n} \leq x + \epsilon\right) + h_N(\epsilon) \\ &\leq \mathbb{P}\left(\sup_{n=1, \dots, \lfloor N/(q+1) \rfloor} Z_{q,n(q+1)} \leq x + \epsilon\right) + h_N(\epsilon) \\ &= \mathbb{P}(Z_{1,q+1} \leq x + \epsilon)^{\lfloor N/(q+1) \rfloor} + h_N(\epsilon), \end{aligned} \quad (5)$$

where we have used the fact that $(Z_{q,n(q+1)})_{n \geq 1}$ is an i.i.d. sequence. Since the (Y_n) are i.i.d. (and therefore exchangeable), we get for $y \in \mathbb{R}$ that

$$\mathbb{P}(Z_{1,q+1} \leq y) = \mathbb{P}\left(\sum_{k=0}^q c_k Y_{k+1} \leq y\right) \rightarrow \mathbb{P}\left(\sum_{k=0}^{\infty} c_k Y_{k+1} \leq y\right), \quad q \rightarrow \infty, \quad (6)$$

since the series $\sum_{k=0}^{\infty} c_k Y_{k+1} =: Z$ converges a.s. by Kolmogorov's Three Series Theorem since $\mathbb{E}[Y_1^2] < \infty$. Next, $\mathbb{P}(Z \leq y) < 1$ for every $0 \leq y < \delta A$ by Theorem 3.7.5 of Lukacs (1970). Then for $0 \leq y < \delta A$, by (6), there is $\rho = \rho(y) < 1$ such that $\mathbb{P}(Z_{1,q+1} \leq y) \leq \rho$ for all q sufficiently large.

Moreover, using first Chebychev's inequality and our assumptions on the sequence $(c_n)_{n \geq 0}$, we obtain that

$$\begin{aligned} h_N(u) &\leq u^{-1} \sum_{n=q+2}^N \sum_{k=q+1}^{n-1} |c_k| \mathbb{E}[|Y_1|] \leq u^{-1} \mathbb{E}[|Y_1|] \sum_{n=q+2}^N \sum_{k=q+1}^{\infty} |c_k| \\ &\leq C N u^{-1} \mathbb{E}[|Y_1|] e^{-\lambda q} = C_1 N u^{-1} e^{-\lambda q}. \end{aligned} \quad (7)$$

Let $q = q_N := \lfloor \beta \sqrt{N} \rfloor$, $\beta > 0$. If $u > 0$ is such that $x + u < A$, we deduce from (5) and (7) that

$$p_N(x) \leq \rho^{\sqrt{N}/\beta} + C_1 N u^{-1} e^{-\lambda \beta \sqrt{N} + \lambda}.$$

By choosing β sufficiently large, the theorem follows under the assumption $\mathbb{E}[Y_1^2] < \infty$. If $\mathbb{E}[\exp(|Y_1|^\alpha)] < \infty$, the estimate on h_N can be improved as follows:

$$\sup_{n=q+1, \dots, N} \left| \sum_{k=q+1}^n c_k Y_k \right| \leq \sup_{l=q+1, \dots, N} |Y_l| \sum_{k=q+1}^{\infty} |c_k| \leq C e^{-\lambda q} \sup_{l=q+1, \dots, N} |Y_l|.$$

Hence,

$$\begin{aligned} h_N(u) &\leq \sum_{n=q+1}^N \mathbb{P} \left(\sup_{k=q+1, \dots, N} |Y_k| > e^{\lambda q} u / C \right) \leq N^2 \mathbb{P}(|Y_1| > e^{\lambda q} u / C) \\ &\leq N^2 \exp(-e^{\alpha \lambda q} (u/C)^\alpha) \mathbb{E}[\exp(|Y_1|^\alpha)]. \end{aligned}$$

In particular, with $q = q_N = \lfloor \kappa \log N \rfloor$, if κ is large enough, this implies together with (5) that, for some $c(x) > 0$,

$$p_N(x) \lesssim N^2 e^{-N^2} + \rho^{\lfloor N/(q_N+1) \rfloor} \lesssim \exp(-c(x)N/\log N), \quad N \rightarrow \infty.$$

□

The proof of Theorem 3.2 reveals that fast decay of p_N be explained intuitively as follows: if we write $X_n = \sum_{k=1}^{n-q-1} c_{n-k} Y_k + \sum_{k=n-q}^n c_{n-k} Y_k$, the first summand is typically small if q is large and $c_n \rightarrow 0$. Hence,

$$\mathbb{P} \left(\sup_{n=1, \dots, N} X_n \leq 0 \right) \approx \mathbb{P} \left(\sup_{n=q+1, \dots, N} \sum_{k=n-q}^n c_{n-k} Y_k \leq 0 \right) \approx \mathbb{P} \left(\sum_{k=1}^{q+1} c_{n-k} Y_k \leq 0 \right)^{N/q}.$$

Remark 3.3. If $(c_k)_{k \geq 0}$ denote a sequence with $c_0 = 1$ and $\sum_{k=0}^{\infty} |c_k| < \infty$ and $|Y_1| \leq M$ a.s. for some $M < \infty$, one can prove in an analogous way that even $p_N \lesssim \exp(-cN)$ for some $c > 0$ since $h_N(u)$ in the proof of Theorem 3.2 vanishes for q large enough..

Remark 3.4. As it was already remarked by Novikov and Kordzakhia (2008), if $(c_k)_{k \geq 0}$ denotes a sequence of positive numbers, one has that

$$X_n = \sum_{k=1}^n c_{n-k} Y_k \geq \sum_{k=1}^n c_{n-k} Y_k 1_{\{Y_k \leq M\}} = \sum_{k=1}^n c_{n-k} \tilde{Y}_k =: \tilde{X}_n,$$

such that $\mathbb{P}(X_n \leq x, \forall n \leq N) \leq \mathbb{P}(\tilde{X}_n \leq x, \forall n \leq N)$. Hence, if the c_n are positive, one can assume without loss of generality that the innovations are bounded from above in order to establish an upper bound on the survival probability.

For AR(2)-processes, Theorem 3.2 is applicable if $a_1 + a_2 < 1$, $a_2 < a_1 + 1$ and $a_2 > -1$, cf. Remark 2.1 and Figure 2.1. Moreover, the preceding theorem can be generalized easily to cover more general processes (such as autoregressive moving average models ARMA(p,q) and moving average processes of infinite order MA(∞), see Section 3 in Brockwell and Davis (1987)):

Theorem 3.5. *Let $(c_k)_{k \in \mathbb{Z}}$ denote a sequence with $c_0 = 1$, $A := \sum_{k=-\infty}^{\infty} |c_k| < \infty$ and $\sum_{|k| \geq q} |c_k| \leq C e^{-\lambda q}$ for all $q \geq 1$ and some $\lambda > 0$. Let $(Y_k)_{k \in \mathbb{Z}}$ be a sequence of i.i.d. random variables such that $\mathbb{E}[Y_1^2] < \infty$ and $\mathbb{P}(Y_1 > \delta) > 0$ and $\mathbb{P}(Y_1 < -\delta) > 0$ for some $\delta > 0$. Let $X_n := \sum_{k=-\infty}^{\infty} c_{n-k} Y_k$ for $n \in \mathbb{Z}$. If $x \in [0, \delta A)$, it holds for some $c(x) > 0$ that*

$$\mathbb{P}\left(\sup_{|n| \leq N} X_n \leq x\right) \lesssim \exp(-c(x)\sqrt{N}), \quad N \rightarrow \infty.$$

Moreover, if $\mathbb{E}[\exp(|Y_1|^\alpha)] < \infty$ ($\alpha > 0$) and $x \in [0, \delta A)$, there is $c(x) > 0$ such that

$$\mathbb{P}\left(\sup_{|n| \leq N} X_n \leq x\right) \lesssim \exp(-c(x)N/\log N), \quad N \rightarrow \infty.$$

Proof. It is well known that X_n is well defined for every $n \in \mathbb{Z}$ under the given assumptions on the sequence (c_n) . The proof is then very similar to that of Theorem 3.2. We define $Z_{q,n} := \sum_{k=n-q}^{n+q} c_{n-k} Y_k$. Note that $(Z_{q,n(2q+1)})_{n \in \mathbb{Z}}$ forms a sequence of i.i.d. random variables with $Z_{q,0} = \sum_{k=-q}^q c_k Y_k$. The remainder of the proof is along the same lines of the proof of Theorem 3.2. \square

In certain special cases, we can improve Theorem 3.2. Namely, if (c_n) is a sequence of positive numbers and $c_n = \rho^n(1 + o(1))$ where $\rho \in (0, 1)$, it follows from Theorem 3.1 that p_N goes to zero exponentially fast under mild assumptions on Y_1 :

Proposition 3.6. *Let $(c_n)_{n \geq 0}$ be a sequence such that $\alpha C \rho^n \leq c_n \leq C \rho^n$ for all $n \geq 0$ where $\rho \in (0, 1)$, $0 < \alpha < 1$, $C > 0$. Assume that $\mathbb{E}[(Y_1^-)^\delta] < \infty$ for some $\delta \in (0, 1)$. Let $x \geq 0$ be such that $\mathbb{P}(Y_1 \geq x(1 - \rho)/(\alpha C)) > 0$. Let $X_n := \sum_{k=1}^n c_{n-k} Y_k$. Then there is some $\lambda = \lambda(x) > 0$ such that $p_N(x) \lesssim \exp(-\lambda N)$.*

Proof. Define the i.i.d. random variables $\tilde{Y}_k := Y_k 1_{\{Y_k < 0\}} + \alpha Y_k 1_{\{Y_k > 0\}}$, $k \geq 1$. Since $c_k \geq 0$ for all k , we obtain that

$$X_n = \sum_{k=1}^n c_{n-k} Y_k \geq \sum_{k=1}^n C \rho^{n-k} Y_k 1_{\{Y_k < 0\}} + \sum_{k=1}^n \alpha C \rho^{n-k} Y_k 1_{\{Y_k > 0\}} = C \sum_{k=1}^n \rho^{n-k} \tilde{Y}_k =: CZ_n,$$

where $Z_n := \rho Z_{n-1} + \tilde{Y}_n$. In particular, we conclude that

$$\mathbb{P} \left(\sup_{n=1, \dots, N} X_n \leq x \right) \leq \mathbb{P} \left(\sup_{n=1, \dots, N} Z_n \leq x/C \right).$$

Now $\mathbb{P}(\tilde{Y}_1 > x(1-\rho)/C) = \mathbb{P}(Y_1 \geq x(1-\rho)/(\alpha C)) > 0$ by the choice of x . Hence, the result follows from Theorem 1 of Novikov and Kordzakhia (2008) (Theorem 3.1 above). \square

The preceding proposition yields the following corollary for AR(2)-processes:

Corollary 3.7. *Let $a_1 \in (0, 2)$, $a_2 < 0$ with $a_1 + a_2 < 1$ and $a_1^2 + 4a_2 > 0$. Assume that $\mathbb{E}[(Y_1^-)^\delta] < \infty$ for some $\delta \in (0, 1)$ and $\mathbb{P}(Y_1 \geq y) > 0$ for every y . For every $x \geq 0$, there is $\lambda = \lambda(x) > 0$ such that $p_N(x) \lesssim \exp(-\lambda N)$.*

Proof. It is not hard to check that $0 < s_2 < s_1 < 1$. Hence, $c_n = s_1^n (s_1 - s_2 (s_2/s_1)^n)/h$ and $h^{-1}(s_1 - s_2)s_1^n \leq c_n \leq h^{-1}s_1^{n+1}$ for all n . The result follows from Proposition 3.6. \square

If $|Y_1| \leq M$ a.s., the preceding corollary is not applicable. However, we already know that $p_N \lesssim e^{-cN}$ for some $c > 0$ in that case, see Remark 3.3.

Let us now establish exponential upper bounds for p_N for certain distributions if the sequence (c_n) oscillates and diverges exponentially. The proof relies on the following proposition.

Proposition 3.8. *Let $\rho \in (-1, 1)$ ($\rho \neq 0$) and set $Z := \sum_{n=1}^{\infty} \rho^n Y_n$. Moreover, suppose that $\mathbb{E}[|Y_1|^a] < \infty$ for some $a > 0$. Let φ denote the characteristic function of Y_1 and assume that there are $\delta \in (0, |\rho|)$ and $t_0 > 0$ such that $|\varphi(t)| \leq \delta$ for all $|t| \geq t_0$. It follows that $\mathbb{P}(|Z| \leq \epsilon) \lesssim \epsilon$ as $\epsilon \downarrow 0$.*

Proof. Z is well-defined and its characteristic function $\tilde{\varphi}$ is given by $\tilde{\varphi}(t) = \prod_{n=1}^{\infty} \varphi(\rho^n t)$, see e.g. Section 3.7 of Lukacs (1970). Let us show that $\tilde{\varphi}$ is absolutely integrable. If this holds, by Theorem 3.2.2 of Lukacs (1970), Z admits a continuous density g which is given by

$$g(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} \tilde{\varphi}(t) dt, \quad x \in \mathbb{R}.$$

In particular, g is bounded implying that $\mathbb{P}(|Z| \leq \epsilon) \leq C\epsilon$ for any $\epsilon \geq 0$.

To prove the integrability of $\tilde{\varphi}$, let δ and t_0 be as in the statement of the proposition and note that

$$|\tilde{\varphi}(t)| = \prod_{n=1}^{\infty} |\varphi(\rho^n t)| \leq \delta^{N(t)},$$

where $N(t) = \#\{n \geq 1 : |\rho^n t| \geq t_0\}$. One verifies that $N(t) = \lfloor (\log(t) - \log(t_0)) / \log(1/|\rho|) \rfloor$ so that

$$|\tilde{\varphi}(t)| \leq \exp \left(\log \delta \left(\frac{\log |t| - \log(t_0)}{\log(1/|\rho|)} - 1 \right) \right) = C |t|^{-\alpha},$$

where C depends on t_0, ρ and δ only and $\alpha := \log(1/\delta) / \log(1/|\rho|) > 1$. This shows that $|\tilde{\varphi}(t)|$ is integrable over \mathbb{R} . \square

Remark 3.9. Recall that if X has an absolutely continuous distribution, it holds that $\lim_{|t| \rightarrow \infty} \mathbb{E}[e^{itX}] = 0$, see e.g. Section 2.2 in Lukacs (1970). However, if the distribution of X is purely discrete, $\limsup_{|t| \rightarrow \infty} |\mathbb{E}[e^{itX}]| = 1$ and in general, it is a very challenging problem to find conditions such that the random series $\sum_{k=1}^{\infty} \rho^n Y_n$ has a density. This question has attracted a lot of attention for so-called infinite Bernoulli convolutions. We refer to the survey of Peres et al. (2000).

We can now prove the following theorem.

Theorem 3.10. *Let $X_n := \sum_{k=1}^n c_{n-k} Y_k$ where $c_n = d\rho^n + \beta_n r^n$ where $d \neq 0$, $\rho < -1$ and $|\rho| > |r|$ and $|\beta_n| e^{-\lambda n} \rightarrow 0$ as $n \rightarrow \infty$ for every $\lambda > 0$. Assume $\mathbb{E}[|Y_1|^a] < \infty$ for some $a > 0$. Moreover, suppose that the characteristic function φ of Y_1 satisfies the inequality $|\varphi(t)| \leq \delta < |\rho|$ for all $|t|$ large enough. Then there is a constant $C > 0$ such that for every $x \geq 0$, it holds that*

$$\liminf_{N \rightarrow \infty} -N^{-1} \log \mathbb{P} \left(\sup_{n=1, \dots, N} X_n \leq x \right) \geq C.$$

If $\mathbb{E}[\exp(|Y_1|^\alpha)] < \infty$ for some $\alpha > 0$, then

$$C \geq \begin{cases} \log |\rho/r|, & |r| > 1, \\ \log |\rho|, & \text{else.} \end{cases}$$

Proof. Assume w.l.o.g. that $d = 1$ (write $X_n = \sum_{k=1}^n (c_{n-k}/d)(dY_k)$). Let $\hat{\beta}_n := \sup\{|\beta_0|, \dots, |\beta_n|\}$ and $E_N := \{|Y_1| \leq f_N, \dots, |Y_N| \leq f_N\}$ where $1 \leq f_N \rightarrow \infty$ is to be specified later. On E_N , it holds for $n = 1, \dots, N$ that

$$\begin{aligned} X_n &= \sum_{k=1}^n c_{n-k} Y_k = \sum_{k=1}^n \rho^{n-k} Y_k + \sum_{k=1}^n \beta_{n-k} r^{n-k} Y_k \\ &\geq \sum_{k=1}^n \rho^{n-k} Y_k - \hat{\beta}_n f_N \sum_{k=1}^n |r|^{n-k} \geq \sum_{k=1}^n \rho^{n-k} Y_k - \hat{\beta}_N f_N \sum_{k=0}^N |r|^k. \end{aligned}$$

Case 1: Consider first the case that $\beta_n \neq 0$ for some n . Let $R_N := \sum_{k=0}^N |r|^k$. Then

$$p_N(x) \leq \mathbb{P}(E_N^c) + \mathbb{P}\left(\sup_{n=1, \dots, N} \sum_{k=1}^n \rho^{n-k} Y_k \leq x + \hat{\beta}_N f_N R_N, E_N\right). \quad (8)$$

Note that $Z_n := \sum_{k=1}^n \rho^{n-k} Y_k$ is an AR(1)-process satisfying $Z_n = \rho Z_{n-1} + Y_n$. Let us begin with the following useful observation: if $Z_{N-1} \leq z$ and $Z_N \leq z$ for some large $z > 0$, we have with high probability that $|Z_{N-1}| \leq z$. This will allow us to reduce the estimation of $p_N(x)$ to controlling $\mathbb{P}(|Z_N| \leq z_N)$ where $z_N \rightarrow \infty$ as $N \rightarrow \infty$. To be precise, note that

$$\begin{aligned} \{Z_{N-1} \leq z, Z_N \leq z\} &\subseteq \{|Z_{N-1}| \leq z\} \cup \{Z_{N-1} < -z, Z_N \leq z\} \\ &\subseteq \{|Z_{N-1}| \leq z\} \cup \{Y_N \leq (1 - |\rho|)z\}. \end{aligned} \quad (9)$$

For the last inclusion, we have used that the event $\{Z_{N-1} < -z, Z_N \leq z\}$ implies that $z \geq Z_N = \rho Z_{N-1} + Y_N \geq -\rho z + Y_N$. Hence, combining this with (8), we obtain that

$$\begin{aligned} p_N(x) &\leq \mathbb{P}(E_N^c) + \mathbb{P}\left(Z_{N-1} \leq x + \hat{\beta}_N f_N R_N, Z_N \leq x + \hat{\beta}_N f_N R_N\right) \\ &\leq \mathbb{P}(E_N^c) + \mathbb{P}\left(|Z_{N-1}| \leq x + \hat{\beta}_N f_N R_N\right) + \mathbb{P}\left(Y_N \leq (1 - |\rho|)(x + \hat{\beta}_N f_N R_N)\right). \end{aligned} \quad (10)$$

It remains to estimate the three probabilities above. Clearly,

$$\mathbb{P}(E_N^c) = \mathbb{P}\left(\bigcup_{n=1}^N \{|Y_n| > f_N\}\right) \leq N \mathbb{P}(|Y_1| > f_N).$$

Next, since $|\rho| > 1$ and $\hat{\beta}_N \geq \beta > 0$ for some $\beta > 0$ and for all $N \geq N_0$ large enough and $R_N \geq 1$, it follows that

$$\mathbb{P}\left(Y_N \leq (1 - |\rho|)(x + \hat{\beta}_N f_N R_N)\right) \leq \mathbb{P}(|Y_1| \geq (|\rho| - 1)\beta f_N), \quad N \geq N_0.$$

For large N , using the last two inequalities in (10), we arrive at

$$p_N(x) \leq (N + 1) \mathbb{P}(|Y_1| \geq C_1 f_N) + \mathbb{P}\left(|Z_{N-1}| \leq 2\hat{\beta}_N f_N R_N\right), \quad (11)$$

where $C_1 := \min\{1, (|\rho| - 1)\beta\}$. Set $\tilde{Z}_n := \rho^{-n} Z_n = \sum_{k=1}^n \rho^{-k} Y_k$. Then

$$\mathbb{P}\left(|Z_{N-1}| \leq 2\hat{\beta}_N f_N R_N\right) = \mathbb{P}\left(|\tilde{Z}_{N-1}| \leq 2|\rho|^{-(N-1)} \hat{\beta}_N f_N R_N\right).$$

Note that \tilde{Z}_n converges a.s. to a random variable \tilde{Z}_∞ by Kolmogorov's Three Series Theorem. Moreover, for $u, v > 0$,

$$\begin{aligned} \mathbb{P}\left(|\tilde{Z}_\infty| \leq u + v\right) &\geq \mathbb{P}\left(|\tilde{Z}_\infty - \tilde{Z}_N| \leq u + v - |\tilde{Z}_N|, |\tilde{Z}_N| \leq u\right) \\ &\geq \mathbb{P}\left(|\tilde{Z}_\infty - \tilde{Z}_N| \leq v, |\tilde{Z}_N| \leq u\right) = \mathbb{P}\left(|\tilde{Z}_\infty - \tilde{Z}_N| \leq v\right) \mathbb{P}\left(|\tilde{Z}_N| \leq u\right). \end{aligned}$$

The last equality follows from the independence of increments of \tilde{Z} . Hence,

$$\mathbb{P}\left(\left|\tilde{Z}_N\right| \leq u\right) \leq \frac{\mathbb{P}\left(\left|\tilde{Z}_\infty\right| \leq u+v\right)}{1 - \mathbb{P}\left(\left|\tilde{Z}_\infty - \tilde{Z}_N\right| > v\right)}, \quad u, v > 0, N \geq 1.$$

Using this inequality with $u = v = C_2 |\rho|^{-N} \hat{\beta}_N f_N R_N$, we obtain that

$$\begin{aligned} \mathbb{P}\left(\left|\tilde{Z}_{N-1}\right| \leq 2 |\rho|^{-(N-1)} \hat{\beta}_N f_N R_N\right) &\leq \frac{\mathbb{P}\left(\left|\tilde{Z}_\infty\right| \leq 4 |\rho|^{-(N-1)} \hat{\beta}_N f_N R_N\right)}{1 - \mathbb{P}\left(\left|\tilde{Z}_\infty - \tilde{Z}_{N-1}\right| > 2 |\rho|^{-(N-1)} \hat{\beta}_N f_N R_N\right)} \\ &\leq 2 \mathbb{P}\left(\left|\tilde{Z}_\infty\right| \leq 4 |\rho|^{-(N-1)} \hat{\beta}_N f_N R_N\right) \end{aligned}$$

where the last inequality holds for all N sufficiently large in view of the following estimates: Since $R_N \geq 1$, $\hat{\beta}_N \geq \beta > 0$ for large N , $\mathbb{E}[|Y_1|^a] < \infty$ (w.l.o.g. $a \in (0, 1)$) and $f_N \rightarrow \infty$, we have that

$$\begin{aligned} \mathbb{P}\left(\left|\tilde{Z}_\infty - \tilde{Z}_{N-1}\right| > 2 |\rho|^{-(N-1)} \hat{\beta}_N f_N R_N\right) &= \mathbb{P}\left(\left|\sum_{n=N}^{\infty} \rho^{-n} Y_n\right|^a > (2\beta |\rho|^{-(N-1)} f_N)^a\right) \\ &\leq \mathbb{P}\left(\sum_{n=N}^{\infty} |\rho|^{-an} |Y_n|^a > (2\beta |\rho|^{-(N-1)} f_N)^a\right) \leq \frac{\sum_{n=N}^{\infty} |\rho|^{-an} \mathbb{E}[|Y_1|^a]}{(2\beta |\rho|^{-(N-1)} f_N)^a} \\ &\leq C_2 \frac{|\rho|^{-aN}}{|\rho|^{-aN} f_N^a} = C_2 \frac{1}{f_N^a} \rightarrow 0. \end{aligned}$$

In the first inequality, we have used that $(x+y)^a \leq x^a + y^a$ for $x, y \geq 0$ and $a \in (0, 1)$. We have shown that (11) implies for all N large enough that

$$p_N(x) \leq (N+1) \mathbb{P}(|Y_1| \geq C_1 f_N) + 2 \mathbb{P}\left(\left|\tilde{Z}_\infty\right| \leq 2C_2 |\rho|^{-N} \hat{\beta}_N f_N R_N\right). \quad (12)$$

If $f_N \rightarrow \infty$ is chosen such that $|\rho|^{-N} \hat{\beta}_N f_N R_N \rightarrow 0$, we conclude from (12) and Proposition 3.8 that

$$p_N(x) \leq (N+1) \mathbb{P}(|Y_1| \geq C_1 f_N) + C_4 |\rho|^{-N} \hat{\beta}_N f_N R_N, \quad N \rightarrow \infty. \quad (13)$$

Let us now state the suitable choice for f_N . First, recall that by assumption, we have that $\hat{\beta}_N = e^{o(N)}$.

Assume first that $|r| \leq 1$. Then $R_N \leq N$. One can set $f_N := \delta^N$ where $1 < \delta < |\rho|$, use Chebychev's inequality (recall that $\mathbb{E}[|Y_1|^a] < \infty$) and (13) to show that

$$p_N(x) \lesssim N \delta^{-aN} + |\rho/\delta|^{-N} e^{o(N)} N = e^{o(N)} (\delta^a \wedge (|\rho|/\delta))^{-N}, \quad N \rightarrow \infty.$$

If $|r| > 1$, $R_N \asymp |r|^N$, take $f_N := \delta^N$ where $1 < \delta < |\rho/r|$, and as above, one sees that

$$p_N(x) \lesssim N\delta^{-a_N} + |\rho/(\delta r)|^{-N} e^{o(N)} = e^{o(N)} (\delta^a \wedge (|\rho/(r\delta)|))^{-N}, \quad N \rightarrow \infty.$$

If $\mathbb{E}[\exp(|Y_1|^\alpha)] < \infty$ for some $\alpha > 0$, it suffices to take $f_N := N^{2/\alpha}$ to obtain

$$p_N(x) \leq (N+1)\mathbb{E}[\exp(|Y_1|^\alpha)] \exp(-C_1^\alpha N^2) + C_3 |\rho|^{-N} e^{o(N)} N^{2/\alpha} R_N,$$

and it is then easy to conclude that $\liminf -N^{-1}p_N(x) \geq -\log(1/|\rho|) = \log(|\rho|)$ if $|r| \leq 1$ and $\liminf -N^{-1}p_N(x) \geq \log(|\rho/r|)$ if $|r| > 1$.

Case 2: Finally, assume that $\beta_n = 0$ for all n . Then $X_n = Z_n = \sum_{k=1}^n \rho^{n-k} Y_k$. Let $0 \leq f_N \rightarrow \infty$ to be specified later. Clearly, for large N ,

$$\begin{aligned} \mathbb{P}\left(\sup_{n=1,\dots,N} Z_n \leq x\right) &\leq \mathbb{P}(Z_{N-1} \leq x, Z_N \leq x) \leq \mathbb{P}(Z_{N-1} \leq f_N, Z_N \leq f_N) \\ &\leq \mathbb{P}(|Z_{N-1}| \leq f_N) + \mathbb{P}(Y_1 \leq (1-|\rho|)f_N), \end{aligned}$$

where we have used (9) in the last inequality. But the last line is just a special case of (10) with $x = 0$, $\hat{\beta}_N = R_N = 1$, so we can proceed as above. \square

We can apply Theorem 3.10 to prove that p_N decays exponentially for $(a_1, a_2) \in E_1$, cf. Figure 2.

Corollary 3.11. *Let $(a_1, a_2) \in E_1$. Assume that Y_1 satisfies the conditions of Theorem 3.10. Then there is a constant $C > 0$ such that for every $x \geq 0$, it holds that*

$$\liminf_{N \rightarrow \infty} -N^{-1} \log \mathbb{P}\left(\sup_{n=1,\dots,N} X_n \leq x\right) \geq C.$$

If $\mathbb{E}[\exp(|Y_1|^\alpha)] < \infty$ for some $\alpha > 0$, then

$$C \geq \begin{cases} \log(|s_2|/s_1), & a_1 + a_2 > 1, \\ \log |s_2|, & \text{else.} \end{cases}$$

Proof. For $(a_1, a_2) \in E_1$, we have that $s_2 < -1$ and $|s_2| > s_1 > 0$. Hence, we can apply Theorem 3.10 with $\rho = s_2$ and $r = s_1$. To get the lower bound on C , note that $|r| = s_1 \leq 1$ amounts to $a_1 + a_2 \leq 1$. \square

Remark 3.12. One can show by direct computation that the correlation coefficient ρ_n of X_{n-1} and X_n , given by

$$\rho_n = \mathbb{E}[X_{n-1}X_n] / \sqrt{\mathbb{E}[X_{n-1}^2] \mathbb{E}[X_n^2]},$$

satisfies $\rho_n = -1 + O(|s_1/s_2|^n)$. Clearly, $p_N \leq \mathbb{P}(X_{n-1} \leq 0, X_n \leq 0)$, and if Y_1 is a centered Gaussian random variable, we get in view of a well-known formula for Gaussian random variables (see e.g. Exercise 8.5.1 in Grimmett and Stirzaker (2001)) that

$$\mathbb{P}(X_{n-1} \leq 0, X_n \leq 0) = \frac{1}{2\pi} \left(\frac{\pi}{2} + \arcsin \rho_n \right).$$

Since $\pi/2 + \arcsin x \sim \sqrt{2(1+x)}$ as $x \downarrow -1$ (by l'Hôpital's rule), it follows that $p_N \lesssim |s_1/s_2|^{N/2}$.

Note that the previous results do not cover the case $a_1 + 1 = a_2$ if $a_2 \in (0, 1)$. Let us now turn to this particular case. One verifies that $c_n = (a_1^{n+1} + (-1)^n)/(a_1 + 1)$, i.e. c_n oscillates but does not diverge as in Theorem 3.10. We show that p_N still decreases at least exponentially in this case.

Proposition 3.13. *Let $a_1 + 1 = a_2$ and set $Z_n = a_2 Z_{n-1} + Y_n$ for $n \geq 1$. Then, for all $x \geq 0$ and $N \geq 1$,*

$$\mathbb{P} \left(\sup_{n=1, \dots, N} X_n \leq x \right) \leq \mathbb{P} \left(\sup_{n=1, \dots, N} Z_n \leq 2x \right).$$

In particular, if $a_2 \in (0, 1)$, $\mathbb{E}[(Y_1^-)^\alpha] < \infty$ and $\mathbb{P}(Y_1 \geq 2x(1 - a_2)) > 0$, it holds that $p_N(x) \lesssim \exp(-\lambda N)$ for some $\lambda = \lambda(x) > 0$.

Proof. Note that $X_{n+1} + X_n = (a_1 + 1)X_n + a_2 X_{n-1} + Y_{n+1} = a_2(X_n + X_{n-1}) + Y_{n+1}$. Hence, $(Z_n)_{n \geq 1}$ can be written in the form $Z_n := X_n + X_{n-1}$. In particular, $X_n \leq x$ for $n = 1, \dots, N$ implies that $Z_n \leq 2x$ for $n = 1, \dots, N$.

If $a_2 \in (0, 1)$, we deduce from Theorem 3.1 that $p_N(x)$ decays exponentially under the conditions stated above. \square

In fact, the idea of proof of Proposition 3.13 can be generalized as follows: if X is AR(p), one can try to determine $b_1, b_2 > 0$ such that $(Z_n)_{n \geq 1}$ is AR($p - 1$) where $Z_n := b_1 X_n + b_2 X_{n-1}$. Then we always have that $X_n \leq 0$ for $n = 1, \dots, N$ implies $Z_n \leq 0$ for $n = 1, \dots, N$. We carry this out for $p = 2$.

Proposition 3.14. *Let $a_1^2 + 4a_2 > 0$. Moreover, assume that either $a_1, a_2 < 0$ or that $a_1 + a_2 < 1$ if $a_2 > 0$. Then $s_2 < 0$, $-a_2/s_2 < 1$ and $Z_n := X_n - s_2 X_{n-1}$ satisfies $Z_n = -a_2/s_2 Z_{n-1} + Y_n$. In particular,*

$$\mathbb{P} \left(\sup_{n=1, \dots, N} X_n \leq x \right) \leq \mathbb{P} \left(\sup_{n=1, \dots, N} Z_n \leq (1 - s_2)x \right), \quad x \geq 0.$$

Proof. Let us determine $b_1, b_2 > 0$ such that $(Z_n)_{n \geq 1}$ defined by $Z_n := b_1 X_n + b_2 X_{n-1}$ is an AR(1)-process. We have that

$$Z_n = (b_1 a_1 + b_2) X_{n-1} + b_1 a_2 X_{n-2} + b_1 Y_n = \frac{b_1 a_1 + b_2}{b_1} b_1 X_{n-1} + \frac{b_1 a_2}{b_2} b_2 X_{n-2} + b_1 Y_n.$$

Hence, if $(b_1 a_1 + b_2)/b_1 = b_1 a_2/b_2$, it follows that

$$Z_n = \frac{b_1 a_2}{b_2} Z_{n-1} + b_1 Y_n = \frac{a_2}{\lambda} Z_{n-1} + b_1 Y_n,$$

where $\lambda := b_2/b_1 > 0$ satisfies $a_1 + \lambda = a_2/\lambda$, i.e. $\lambda^2 + a_1\lambda - a_2 = 0$. The solutions to this equation are $-s_1$ and $-s_2$. Since $a_1^2 + 4a_2 > 0$, we have that $s_2 < s_1$. Hence, we can find $\lambda > 0$ such that Z defines an AR(1)-process if and only if $s_2 < 0$, and $\lambda = -s_2$ in that case. Now $s_2 < 0$ amounts to $a_1 \leq 0$ or $a_1, a_2 > 0$ since $h > 0$.

It follows that

$$\bigcap_{n=1}^N \{X_n \leq x\} \subseteq \bigcap_{n=1}^N \{Z_n \leq (b_1 + b_2)x\} = \bigcap_{n=1}^N \{Z_n \leq b_1(1 - s_2)x\}, \quad x \geq 0.$$

Finally, $a_2/\lambda < 1$ if and only if $a_1 + 2a_2 < h$. If $a_1, a_2 > 0$, this amounts to $a_1 + a_2 < 1$. In the remaining cases, we necessarily have that $a_1 \leq 0$. If also $a_1 + 2a_2 \leq 0$ (in particular, if $a_1, a_2 \leq 0$), the inequality is obviously satisfied. Finally, if $a_1 + 2a_2 > 0$, $a_1 + 2a_2 < h$ is equivalent to $a_1^2 + 4a_1a_2 + 4a_2^2 < a_1^2 + 4a_2$, i.e. $a_1 + a_2 < 1$ since $a_2 > 0$. The assertion of the proposition follows if we set $b_1 = 1$ and $b_2 = -s_2$. \square

The preceding proposition allows us to find exponential upper bounds for the survival probability p_N for a wide class of distributions. Specifically, we obtain exponential upper bounds for certain parameters a_1 and a_2 and distributions that do not fulfill the requirements of Theorem 3.10. Let us record this result as a corollary:

Corollary 3.15. *Let a_1, a_2 be such that $a_2 > 0$ and $a_1 + a_2 < 1$. Assume that $\mathbb{E}[(Y_1^-)^\alpha] < \infty$ for some $\alpha > 0$. Let $x \geq 0$ such that $\mathbb{P}(Y_1 > x(1 - s_2)(1 - a_2/s_2)) > 0$. Then $p_N(x) \lesssim \exp(-\lambda N)$ for some $\lambda = \lambda(x) > 0$.*

Proof. Set $\rho := -a_2/s_2$ and let $(Z_n)_{n \geq 1}$ satisfy $Z_n = \rho Z_{n-1} + Y_n$. By Proposition 3.14, we have that $\rho \in (0, 1)$ and that $p_N(x) \leq \mathbb{P}(\sup_{n=1, \dots, N} Z_n \leq x(1 - s_2))$. The claim now follows from Theorem 3.1. \square

Let us finally turn to the region $a_1 > 0$ and $a_1^2 + 4a_2 < 0$ (E_3 in Figure 2) so that the sequence c_n involves expressions with sine and cosine, cf. (3).

Proposition 3.16. *Let $(a_1, a_2) \in E_3$. Assume that $\mathbb{P}(Y_1 > 0) > 0$. Then there exists $\lambda > 0$ such that $p_N \lesssim \exp(-\lambda N)$ as $N \rightarrow \infty$.*

Proof. The recursion $X_n = a_1 X_{n-1} + a_2 X_{n-2} + Y_n$ allows us to express X_n as follows ($n \geq k + 2$):

$$X_n = \alpha_k X_{n-k} + \beta_k X_{n-k-1} + L_k(Y_{n-k+1}, \dots, Y_n)$$

where $L_k(x_1, \dots, x_k)$ is some linear combination of x_1, \dots, x_k . Clearly, $\alpha_1 = a_1$, $\beta_1 = a_2$ and $L_1(x_1) = x_1$ and iteratively, we get that $\alpha_{k+1} = a_1 \alpha_k + \beta_k$, $\beta_{k+1} = a_2 \alpha_k$ and $L_{k+1}(x_1, \dots, x_{k+1}) = \alpha_k x_1 + L_k(x_2, \dots, x_{k+1})$ for $k \geq 1$. In particular, $\alpha_k = a_1 \alpha_{k-1} + a_2 \alpha_{k-2}$ for $k \geq 2$ with $\alpha_0 = 1$ and $\alpha_1 = a_1$, hence,

$$\alpha_k = c_k, \quad \beta_k = a_2 c_{k-1}, \quad L_k(x_1, \dots, x_k) = \sum_{j=1}^k c_{k-j} x_j.$$

Let $q := \inf \{k \geq 1 : c_k \leq 0\}$. Assume that $q < \infty$ by (3). Then, if $X_n \leq 0$ for all $n \leq N$, it follows that

$$\begin{aligned} 0 &\geq X_n = c_q X_{n-q} + a_2 c_{q-1} X_{n-q-1} + L_q(Y_{n-q+1}, \dots, Y_n) \\ &\geq 0 + 0 + L_q(Y_{n-q+1}, \dots, Y_n), \quad n = q+2, \dots, N, \end{aligned}$$

where we have used the fact that $a_2 c_{q-1} < 0$ by the definition of q . In particular, we have that

$$\begin{aligned} \mathbb{P} \left(\sup_{n=1, \dots, N} X_n \leq 0 \right) &\leq \mathbb{P} \left(\sup_{n=q+2, \dots, N} L_q(Y_{n-q+1}, \dots, Y_n) \leq 0 \right) \\ &\leq \mathbb{P} \left(\sup_{k=1, \dots, \lfloor N/(q+1) \rfloor} L_q(Y_{k(q+1)-q+1}, \dots, Y_{k(q+1)}) \leq 0 \right) \\ &\leq \mathbb{P} (L_q(Y_2, \dots, Y_{q+1}) \leq 0)^{\lfloor N/(q+1) \rfloor}, \end{aligned}$$

since $(L_q(Y_{kq+1}, \dots, Y_{(k+1)q}))_{k=0,1,\dots}$ are i.i.d. Next, note that X_q and $L_q(Y_2, \dots, Y_{q+1})$ have the same law. Hence, using that $c_0, \dots, c_{q-1} > 0$ and $\mathbb{P}(Y_1 > 0) > 0$, we have that

$$\mathbb{P}(X_q > 0) = \mathbb{P} \left(\sum_{k=1}^q c_{q-k} Y_k > 0 \right) \geq \mathbb{P}(Y_1 > 0)^q > 0.$$

It remains to show that $q < \infty$. Let $\varphi \in (0, \pi/2)$ be the angle associated with (a_1, a_2) in (3). Since $a_1 > 0$, it follows from (3) that $c_n \leq 0$ for some n if $\sin(n\varphi) \leq 0$ and $\cos(n\varphi) \leq 0$ for some n . Take $n = \lceil \pi/\varphi \rceil$. Clearly, $\pi \leq n\varphi \leq (\pi/\varphi + 1)\varphi \leq 3\pi/2$ since $\varphi \leq \pi/2$. Since $\sin x \leq 0$ and $\cos x \leq 0$ for all $x \in [\pi, 3\pi/2]$, we have shown that $q \leq \lceil \pi/\varphi \rceil$. \square

We are now ready to give a proof of Theorem 1.2 which is a corollary of the previous results. A look at Figure 2 will be helpful to distinguish the different cases.

Proof. (of Theorem 1.2) On E_1 , the assertion follows from Corollary 3.11. On $E_2 = (-\infty, 0]^2$, the assertion is trivial. If $(a_1, a_2) \in E_3$, we can apply Proposition 3.16. The remaining cases covered by Theorem 3.2 and Proposition 3.13 (the latter is needed for the strip $a_2 = 1 + a_1$ with $a_1 \in (-1, 0)$ only). \square

Note that we have established exponential upper bounds on p_N under various conditions on the distribution of Y_1 in the region where c_n goes to 0 for AR(2)-processes (cf. Remark 2.1) except for the curve $a_1^2 + 4a_2 = 0$ where $a_1 \in (-2, 2)$ and $c_n = (a_1/2)^n(n+1)$. By Theorem 3.2, we know that $p_N \lesssim \exp(-\lambda N/\log N)$ in that case if $\mathbb{E}[\exp(|Y_1|^\alpha)]$ is finite. If Y_1 has a Gaussian law with zero mean, the next proposition establishes an exponential upper bound on p_N in that case. In particular, in combination with the Theorems 1.1, 1.2 and 1.3, we directly obtain Theorem 1.4.

Proposition 3.17. *Let Y_1 have a Gaussian law. Let $\rho \in (0, 1)$ and $(\alpha_n)_{n \geq 0}$ denote a sequence of positive numbers with the following properties*

$$\alpha_{n+m} \leq C \alpha_n \alpha_m \quad (n, m \geq 0), \quad \lim_{n \rightarrow \infty} e^{-\lambda n} \alpha_n = 0 \quad \forall \lambda > 0.$$

Set $X_n := \sum_{k=1}^n \alpha_{n-k} \rho^{n-k} Y_k$. It holds that

$$\liminf_{N \rightarrow \infty} -N^{-1} \log \mathbb{P} \left(\sup_{n=1, \dots, N} X_n \leq x \right) > 0, \quad x \in \mathbb{R}.$$

Proof. Clearly, we may suppose that $\mathbb{E}[(Y_1 - \mathbb{E}[Y_1])^2] = 1$. Moreover, it suffices to consider the case $\mathbb{E}[Y_1] = 0$. To see this, set $\sum_{k=1}^n \alpha_{n-k} \rho^{n-k} (Y_k - \mu)$. If $\mu := \mathbb{E}[Y_1] < 0$, we have that

$$X_n = \sum_{k=1}^n \alpha_{n-k} \rho^{n-k} (Y_k - \mu) + \mu \sum_{k=0}^{n-1} \alpha_k \rho^k \geq \tilde{X}_n + \mu \sum_{k=0}^{\infty} \alpha_k \rho^k,$$

where $A := \sum_{k=0}^{\infty} \alpha_k \rho^k < \infty$ since $\rho < 1$ and $\alpha_n = e^{o(n)}$. Hence,

$$\mathbb{P} \left(\sup_{n=1, \dots, N} X_n \leq x \right) \leq \mathbb{P} \left(\sup_{n=1, \dots, N} \tilde{X}_n \leq x - \mu A \right).$$

Similarly, if $\mu > 0$, $X_n \geq \tilde{X}_n$ for all n , and therefore

$$\mathbb{P} \left(\sup_{n=1, \dots, N} X_n \leq x \right) \leq \mathbb{P} \left(\sup_{n=1, \dots, N} \tilde{X}_n \leq x \right).$$

Hence, we can assume from now on that $\mathbb{E}[Y_1] = 0$ and $\mathbb{E}[Y_1^2] = 1$. Let $\rho < \delta < 1$ and set

$$\gamma_n := \sqrt{\frac{\sum_{k=0}^{n-1} \rho^{2k} \alpha_k^2}{\sum_{k=0}^{n-1} \delta^{2k}}}, \quad Z_n := \gamma_n \sum_{k=1}^n \delta^{n-k} Y_k.$$

We would like to apply Slepian's inequality (Corollary 3.12 in Ledoux and Talagrand (1991)) to compare the probabilities that X and Z stay below 0 until time N . By construction, we have that $\mathbb{E}[X_n^2] = \mathbb{E}[Z_n^2]$ for all $n \geq 1$. Next, note that $\gamma_n \geq \alpha_0 \sqrt{1 - \delta^2}$ for all $n \geq 1$. Hence, if $n > m \geq 1$, we have that

$$\mathbb{E}[Z_n Z_m] = \gamma_n \gamma_m \sum_{k=1}^m \delta^{n-k} \delta^{m-k} \geq \alpha_0^2 (1 - \delta^2) \delta^{n-m} \sum_{k=1}^m \delta^{2(m-k)} \geq C_1 \delta^{n-m},$$

where $C_1 := \alpha_0^2 (1 - \delta^2)$. Moreover,

$$\begin{aligned} \mathbb{E}[X_n X_m] &= \sum_{k=1}^m \alpha_{n-k} \alpha_{m-k} \rho^{m-k} \rho^{n-k} = \rho^{n-m} \sum_{k=1}^m \alpha_{(n-m)+m-k} \alpha_{m-k} \rho^{2(m-k)} \\ &\leq C \rho^{n-m} \alpha_{n-m} \sum_{k=1}^m \alpha_{m-k}^2 \rho^{2(m-k)} \leq C \rho^{n-m} \alpha_{n-m} \sum_{k=0}^{\infty} \alpha_k^2 \rho^{2k} =: C_2 \rho^{n-m} \alpha_{n-m}. \end{aligned}$$

In the last equality, we have used that $\sum_{k=0}^{\infty} \alpha_k^2 \rho^{2k}$ converges since $\alpha_n = e^{o(n)}$. Now $C_1 \delta^{n-m} \geq C_2 \alpha_{n-m} \rho^{n-m}$ holds whenever $n - m \geq q$ for some $q \geq 1$ since $\delta > \rho$ and a_n grows slower than any exponential. In particular, $\mathbb{E}[X_n X_m] \leq \mathbb{E}[Z_n Z_m]$ whenever $|n - m| \geq q$.

Hence, using Slepian's inequality, we obtain that

$$\mathbb{P} \left(\sup_{n=1, \dots, N} X_n \leq x \right) \leq \mathbb{P} \left(\sup_{n=1, \dots, \lfloor N/q \rfloor} X_{nq} \leq x \right) \leq \mathbb{P} \left(\sup_{n=1, \dots, \lfloor N/q \rfloor} Z_{nq} \leq x \right).$$

Let $\tilde{Z}_n := \delta^{-nq} Z_{nq} / \gamma_{nq} = \sum_{k=1}^{nq} \delta^{-k} Y_k$. One verifies easily that $(\tilde{Z}_n)_{n \geq 1}$ is equal in distribution to $(B(t_n))_{n \geq 1}$ where $(B_t)_{t \geq 0}$ is a one-dimensional Brownian motion and $t_n := \sum_{k=1}^{nq} \delta^{-2k} = C_\delta (\delta^{-2nq} - 1)$, so

$$\begin{aligned} p_N &\leq \mathbb{P} \left(\sup_{n=1, \dots, N} Z_{nq} \leq x \right) = \mathbb{P} \left(\bigcap_{n=1}^N \left\{ \tilde{Z}_n \leq x \delta^{-nq} / \gamma_{nq} \right\} \right) \\ &= \mathbb{P} \left(\bigcap_{n=1}^N \left\{ B(C_\delta (\delta^{-2nq} - 1)) \leq x \delta^{-nq} / \gamma_{nq} \right\} \right) \leq \mathbb{P} \left(\sup_{n=1, \dots, N} B(\delta^{-2nq} - 1) \leq \tilde{x} \right), \end{aligned}$$

where we have used the scaling property of Brownian motion and the fact that $\gamma_n \geq C_1 \alpha_0 / \delta^n$ for all n (i.e. $\tilde{x} := x / (C_1 \alpha_0 C_\delta^{1/2})$). Next, note that

$$\begin{aligned} \mathbb{P} \left(\sup_{n=1, \dots, N} B(\delta^{-2nq} - 1) \leq 0 \right) &\geq \mathbb{P} \left(B_1 \leq -\tilde{x}, \sup_{n=1, \dots, N} B(\delta^{-2nq} - 1) - B_1 \leq \tilde{x} \right) \\ &= \mathbb{P}(B_1 \leq -\tilde{x}) \mathbb{P} \left(\sup_{n=1, \dots, N} B(\delta^{-2nq} - 1) \leq \tilde{x} \right). \end{aligned}$$

An application of Slepian's inequality together with a subadditivity argument (see e.g. Eq. 2.6 of Aurzada and Baumgarten (2011)) yields that

$$\liminf_{N \rightarrow \infty} N^{-1} \log \mathbb{P} \left(\sup_{n=1, \dots, N} B(a^n) \leq 0 \right) > 0, \quad a > 1.$$

□

3.2 Exponential lower bounds

Let us now comment on exponential lower bounds for AR-processes. In general, we cannot expect to find exponential lower bounds in the whole region where we have established exponential upper bounds. The following example illustrates this point for AR(2)-processes.

Example 3.18. If X is AR(p) and the innovation Y_1 takes only the values $\pm y$ for some $y > 0$ and $a_1 < -1$, then $p_2 = \mathbb{P}(X_1 \leq 0, X_2 \leq 0) = 0$. Indeed, on $\{X_1 \leq 0\} = \{Y_1 = -y\}$, we have that $X_2 = a_1 Y_1 + Y_2 \geq -y a_1 - y = -y(a_1 + 1) > 0$. Similarly, if $a_1 \in [-1, 0]$ and $a_1(a_1 + 1) + a_2 < -1$, one has that $p_3 = 0$.

Let us also remark that if X is $AR(p)$ with $a_1 \geq 0, \dots, a_p \geq 0$, it is trivial to obtain the exponential lower bound $p_N(x) \geq p_N \geq \mathbb{P}(Y_1 \leq 0)^N$.

The following theorem states a simple condition on the coefficients a_1, \dots, a_p such that the survival probability cannot decay faster than exponentially.

Theorem 3.19. *If X is $AR(p)$ with $\sum_{k=1}^p |a_k| < 1$, it holds that $p_N \gtrsim c^N$ for all N where $c \in (0, 1)$. Moreover, if $a_k > 0$ for some $k \in \{1, \dots, p\}$, one may take*

$$c := \sup \{ \mathbb{P}(Y_1 \in [\alpha(1 - a_+), \alpha |a_-|]) : \alpha < 0 \}$$

where (with the convention that $\sum_{\emptyset} = 0$)

$$a_+ := \sum_{k \in I_+} a_k, \quad a_- := \sum_{k \in I_-} a_k, \quad I_+ = \{k : a_k > 0\}, \quad I_- = \{k : a_k < 0\}.$$

Proof. The goal is to find intervals $([\alpha_n, \beta_n])_{n \geq 1}$ such that

$$\bigcap_{k=1}^n \{Y_k \in [\alpha_k, \beta_k]\} \subseteq \bigcap_{k=1}^n \{X_k \in [\gamma_k, 0]\}, \quad n \geq 1. \quad (14)$$

If (14) holds and $\mathbb{P}(Y_n \in [\alpha_n, \beta_n]) \geq c > 0$ for all $n \geq N_0$, we immediately obtain that $p_N \gtrsim c^N$.

Using the recursive definition of X , we can iteratively define the sequences $(\alpha_n)_{n \geq 1}$, $(\beta_n)_{n \geq 1}$, $(\gamma_n)_{n \geq 1}$ as follows: Start with $\gamma_1 = \alpha_1 < \beta_1 \leq 0$. Define successively (with the convention $\gamma_n = 0$ for $n \leq 0$)

$$\beta_k := - \sum_{j \in I_-} a_j \gamma_{k-j}, \quad \alpha_k < \beta_k, \quad \gamma_k := \sum_{j \in I_+} a_j \gamma_{k-j} + \alpha_k.$$

It is clear that $\gamma_k \leq 0$ and $\beta_k \leq 0$ for all k . We claim that (14) holds for this choice of (α_n) , (β_n) and (γ_n) . For $n = 1$, this is obvious, and inductively, if the statement holds for some $n - 1 \geq 1$, we have that

$$X_n = \sum_{j=1}^p a_k X_{n-j} + Y_n \leq \sum_{j \in I_-} a_j X_{n-j} + \beta_n \leq \sum_{j \in I_-} a_j \gamma_{n-j} + \beta_n = 0,$$

and

$$X_n = \sum_{j=1}^p a_k X_{n-j} + Y_n \geq \sum_{j \in I_+} a_j X_{n-j} + \alpha_n \geq \sum_{j \in I_+} a_j \gamma_{n-j} + \alpha_n = \gamma_n.$$

Note that the above inequalities hold even if $I_+ = \emptyset$ or if $I_- = \emptyset$. Fix $\alpha_1 = \gamma_1 < \beta_1 = 0$ and let $\alpha_k = -\alpha_1(a_+ - 1)$ for all $k \geq 2$. We claim that $\gamma_k \geq \alpha_1$. Inductively, if the claim holds for all $k \leq n - 1$, we have that

$$\gamma_n = \sum_{j \in I_+} a_j \gamma_{n-j} - \alpha_1(a_+ - 1) \geq \alpha_1 a_+ - \alpha_1(a_+ - 1) = \alpha_1.$$

It follows that $\beta_n \geq -\alpha_1 a_-$ and in particular, $\alpha_k < \beta_k$ since

$$\alpha_k - \beta_k \leq -\alpha_1(a_+ - 1) + \alpha_1 a_- = -\alpha_1 \left(\sum_{k=1}^p |a_k| - 1 \right) < 0.$$

In view of (14), we obtain that

$$\begin{aligned} \mathbb{P} \left(\sup_{k=1, \dots, n} X_k \leq 0 \right) &\geq \prod_{k=1}^n \mathbb{P}(Y_k \in [\alpha_k, \beta_k]) \\ &\geq \mathbb{P}(Y_1 \in [\alpha_1, 0]) \mathbb{P}(Y_1 \in [-\alpha_1(a_+ - 1), -\alpha_1 a_-])^{n-1} \end{aligned}$$

□

Remark 3.20. In general, there is no reason to believe that the lower bound of Theorem 3.19 is sharp.

Corollary 3.21. *Let $(Y_n)_{n \geq 0}$ be a sequence of i.i.d. standard Gaussian random variables. Using the notation of Theorem 3.19, if I_- and I_+ are nonempty, we have that $p_N \geq c^N$ where*

$$c = \mathbb{P}(\alpha^*(1 - a_+) \leq Y_1 \leq \alpha^* |a_-|) = \mathbb{P} \left(-\sqrt{\frac{-\log A^2}{1 - A^2}} \leq Y_1 \leq -A \sqrt{\frac{-\log A^2}{1 - A^2}} \right)$$

and

$$\alpha^* := -\sqrt{\frac{\log(1 - a_+)^2 - \log |a_-|^2}{(1 - a_+)^2 - |a_-|^2}} < 0, \quad A := \frac{|a_-|}{1 - a_+} \in (0, 1).$$

Proof. By Theorem 3.19, we have to determine

$$\sup_{\alpha \leq 0} \mathbb{P}(\alpha(1 - a_+) \leq Y_1 \leq \alpha |a_-|) = \sup_{\alpha \leq 0} \{\Phi(\alpha |a_-|) - \Phi(\alpha(1 - a_+))\},$$

where Φ is the cdf of a standard normal random variable. It is not hard to verify that the unique maximum is attained at

$$\alpha^* := -\sqrt{\frac{\log(1 - a_+)^2 - \log |a_-|^2}{(1 - a_+)^2 - |a_-|^2}} < 0.$$

□

4 Polynomial order

If X is an AR(2)-process and $\mathbb{E}[Y_1] = 0$, it is known that p_N decays polynomially if X is a centered random walk ($a_1 = 1, a_2 = 0$) or an integrated random walk ($a_1 = 2, a_2 = -1$) under suitable moment conditions. To be more precise, if $S_n = \sum_{k=1}^n Y_k$ is a random walk and $\mathbb{E}[Y_1] = 0$, it holds that

$$\mathbb{P}\left(\sup_{n=1,\dots,N} S_n \leq 0\right) = N^{-(1-\theta)+o(1)} \quad \text{for some } \theta \in (0, 1) \iff \mathbb{P}(S_N \leq 0) \rightarrow \theta \in (0, 1),$$

see e.g. Aurzada and Simon (2012). Moreover, the process $X_n = 2X_{n-1} - X_{n-2} + Y_n$ is given by $X_n = \sum_{k=1}^n (n-k+1)Y_k = \sum_{k=1}^n S_k$ where $(S_n)_{n \geq 1}$ is the usual random walk. X is called integrated random walk (IRW). Several authors have studied the asymptotic behaviour of p_N in that case if $\mathbb{E}[Y_1] = 0$. We refer to the recent article of Dembo et al. (2012) and the references therein. In particular, it is shown in Dembo et al. (2012) that $p_N \asymp N^{-1/4}$ if $\mathbb{E}[Y_1] = 0$ and $\mathbb{E}[Y_1^2] \in (0, \infty)$.

4.1 Integrated processes

In this subsection, we will prove that $p_N = N^{-1/2+o(1)}$ under suitable moment conditions if $a_1 + a_2 = 1$ and $|a_2| < 1$. As we will see shortly, these AR(2)-processes can be written as integrated AR(1)-processes.

Let us begin by characterizing the behaviour of the sequence $(c_n)_{n \geq 0}$ for such a_1, a_2 . Instead of manipulating the explicit expression for c_n to determine these values of a_1 and a_2 , we give a short proof of the following lemma.

Lemma 4.1. *The sequence (c_n) converges to a constant $c \neq 0$ if and only if $a_1 + a_2 = 1$ and $|a_2| < 1$. In that case, $\lim_{n \rightarrow \infty} c_n = 1/(1 + a_2)$. Moreover, if $a_1 + a_2 = 1$, $c_n = (1 - (-a_2)^{n+1})/(1 + a_2)$ if $a_2 \neq -1$ and $c_n = n + 1$ if $a_2 = -1$ for $n \geq 0$.*

Proof. Assume that $a_1 + a_2 = 1$. Then $c_{n+1} = (a_1 + a_2 - a_2)c_n + a_2 c_{n-1} = c_n - a_2(c_n - c_{n-1})$, i.e. $c_{n+1} - c_n = -a_2(c_n - c_{n-1})$. Iteration yields $c_{n+1} - c_n = (-a_2)^n(c_1 - c_0) = (-a_2)^{n+1}$. Hence,

$$c_n = 1 + \sum_{k=1}^n (c_k - c_{k-1}) = \begin{cases} 1 + \sum_{k=1}^n (-a_2)^k = \frac{1 - (-a_2)^{n+1}}{1 - (-a_2)}, & a_2 \neq -1, \\ n + 1, & a_2 = -1, \end{cases}$$

and therefore, $c_n \rightarrow c = 1/(1 + a_2) \neq 0$ if and only if $|a_2| < 1$. On the other hand, if $\lim c_n = c \neq 0$, then the recursion equation implies that $c = a_1 c + a_2 c$, i.e. $a_1 + a_2 = 1$. By the preceding lines, convergence implies that $|a_2| < 1$. \square

In particular, the preceding lemma shows that

$$X_n = \frac{1}{1 + a_2} \left(\sum_{k=1}^n Y_k - \sum_{k=1}^n (-a_2)^{n-k+1} Y_k \right), \quad n \geq 1,$$

and since $|a_2| < 1$, one expects that the behaviour of X is similar to that of a random walk.

Moreover, AR(2)-processes with $a_1 + a_2 = 1$ and $|a_2| < 1$ can also be regarded as integrated AR(1)-processes. Let us explain this in more detail.

If \tilde{X} is AR(p) with coefficients a_1, \dots, a_p , set $X_n := \sum_{k=1}^n \tilde{X}_k$.

$$\begin{aligned} X_n &= X_{n-1} + \sum_{k=1}^p a_k \tilde{X}_{n-k} + Y_n = X_{n-1} + \sum_{k=1}^p a_k (X_{n-k} - X_{n-k-1}) + Y_n \\ &= (1 + a_1)X_{n-1} + \sum_{k=2}^p (a_k - a_{k-1})X_{n-k} - a_p X_{n-p-1} + Y_n, \end{aligned}$$

i.e. X is AR($p+1$) and the transformation of the coefficients $T_p: \mathbb{R}^p \rightarrow \mathbb{R}^{p+1}$ is given by

$$T_p(a_1, \dots, a_p) = (a_1 + 1, a_2 - a_1, \dots, a_p - a_{p-1}, -a_p). \quad (15)$$

Note that T_p is one-to-one and that $T_p(\mathbb{R}^p)$ is an affine subspace of \mathbb{R}^{p+1} .

Now, if \tilde{X} is AR(1) with $\tilde{X}_n = \rho \tilde{X}_{n-1} + Y_n$, we have that X with $X_n = \sum_{k=1}^n \tilde{X}_k$ is AR(2) with coefficients $T_1(\rho) = (\rho - 1, -\rho) =: (a_1, a_2)$. In other words, AR(2)-processes with $a_1 + a_2 = 1$ and $|a_2| < 1$ are integrated AR(1)-processes with $|\rho| < 1$.

The next theorem states conditions under which the survival probability of an integrated process behaves like $N^{-1/2+o(1)}$.

Theorem 4.2. *Assume that $\mathbb{E}[Y_1] = 0$. Let $\tilde{X}_n = \sum_{k=1}^n \tilde{c}_{n-k} Y_k$ where $\sum_{k=1}^\infty k |\tilde{c}_k| < \infty$ and $\sum_{k=0}^\infty \tilde{c}_k \neq 0$. Set $X_n := \sum_{k=1}^n \tilde{X}_k$.*

1. *If $|Y_1| \leq M < \infty$ a.s., there is $x_0 \geq 0$ such that for all $x \geq x_0$, it holds that*

$$\mathbb{P} \left(\sup_{n=1, \dots, N} X_n \leq x \right) \asymp N^{-1/2}, \quad N \rightarrow \infty.$$

2. *If $\mathbb{E}[\exp(|Y_1|^\alpha)] < \infty$, it holds for all $x \geq 0$ that*

$$\mathbb{P} \left(\sup_{n=1, \dots, N} X_n \leq x \right) \lesssim N^{-1/2} (\log N)^{1/\alpha}, \quad N \rightarrow \infty.$$

3. *If $\mathbb{E}[\exp(|Y_1|^\alpha)] < \infty$ and $\sum_{k=0}^n \tilde{c}_k \geq 0$ for all $n \geq 0$, it holds for all $x \geq 0$ that*

$$\mathbb{P} \left(\sup_{n=1, \dots, N} X_n \leq x \right) \lesssim N^{-1/2} (\log N)^{-1/\alpha+o(1)}, \quad N \rightarrow \infty.$$

Proof. First, note that

$$X_n = \sum_{k=1}^n \sum_{j=1}^k \tilde{c}_{k-j} Y_j = \sum_{j=1}^n Y_j \sum_{k=j}^n \tilde{c}_{k-j} = \sum_{j=1}^n Y_j \sum_{k=0}^{n-j} \tilde{c}_k = \sum_{k=1}^n c_{n-k} Y_k$$

where $c_n := \sum_{k=0}^n \tilde{c}_k \rightarrow c = \sum_{k=0}^{\infty} \tilde{c}_k \neq 0$. Set $S_n := \sum_{k=1}^n cY_k$, so that for all $n \geq 1$,

$$|S_n - X_n| = \left| \sum_{k=1}^n (c - c_{n-k})Y_k \right|,$$

In particular, if $|Y_1| \leq M < \infty$ a.s., it follows that

$$|S_n - X_n| \leq M \sum_{k=0}^{n-1} |c - c_k| \leq M \sum_{k=0}^{n-1} \sum_{j=k+1}^{\infty} |\tilde{c}_j| = M \sum_{j=1}^{\infty} j |\tilde{c}_j| =: \tilde{M} < \infty.$$

Hence, we get for $x \geq \tilde{M}$ that

$$\mathbb{P} \left(\sup_{n=1, \dots, N} S_n \leq 0 \right) \leq \mathbb{P} \left(\sup_{n=1, \dots, N} X_n \leq x \right) \leq \mathbb{P} \left(\sup_{n=1, \dots, N} S_n \leq x + \tilde{M} \right),$$

and the proof of part 1. is complete since S is a centered random walk with finite variance.

The proof of part 2. is similar. Let $E_N := \{|Y_k| \leq (2 \log N)^{1/\alpha}, k = 1, \dots, N\}$. On E_N , we get as above that

$$|S_n - X_n| \leq (2 \log N)^{1/\alpha} \sum_{k=0}^{n-1} |c - c_k| \leq (2 \log N)^{1/\alpha} \sum_{j=1}^{\infty} j |\tilde{c}_j| =: C(\log N)^{1/\alpha}. \quad (16)$$

Hence,

$$\mathbb{P} \left(\sup_{n=1, \dots, N} X_n \leq x \right) \leq \mathbb{P}(E_N^c) + \mathbb{P} \left(\sup_{n=1, \dots, N} S_n \leq x + C(\log N)^{1/\alpha} \right).$$

By Chebyshev's inequality,

$$\mathbb{P}(E_N^c) \leq N \mathbb{P}(|Y_1| \geq (2 \log N)^{1/\alpha}) \leq N \mathbb{E}[\exp(|Y_1|^\alpha)] N^{-2} \asymp N^{-1}.$$

Finally, by Lemma 4.4 below, it holds that

$$\mathbb{P} \left(\sup_{n=1, \dots, N} S_n \leq x + C(\log N)^{1/\alpha} \right) \lesssim (\log N)^{1/\alpha} N^{-1/2},$$

which proves part 2.

It suffices to prove the lower bound of part 3 for $x = 0$. Moreover, we use that independent random variables Y_1, \dots, Y_N are associated for every N , cf. Esary et al. (1967). Since $c_n = \sum_{k=0}^n \tilde{c}_k \geq 0$ for every n by assumption, the function

$$f_{K,L}(x_1, \dots, x_N) \mapsto \begin{cases} -1, & \sum_{k=1}^n c_{n-k} x_k \leq 0 \text{ for all } n = K, \dots, L \\ 0, & \text{else,} \end{cases}$$

is nondecreasing in every component. Hence, the very definition of associated random variables implies for $1 \leq N_0 < N$ that

$$\text{cov}(f_{1,N_0}(Y_1, \dots, Y_N), f_{N_0+1,N}(Y_1, \dots, Y_N)) \geq 0,$$

or equivalently,

$$\mathbb{P}\left(\sup_{n=1,\dots,N} X_n \leq 0\right) \geq \mathbb{P}\left(\sup_{n=1,\dots,N_0} X_n \leq 0\right) \mathbb{P}\left(\sup_{n=N_0+1,\dots,N} X_n \leq 0\right).$$

Hence, we can bound the survival probability p_N of X from below as follows:

$$\begin{aligned} p_N &\geq p_{N_0} \cdot \mathbb{P}\left(\sup_{n=N_0+1,\dots,N} X_n \leq 0, E_N\right) \\ &\geq p_{N_0} \cdot \mathbb{P}\left(\sup_{n=N_0+1,\dots,N} S_n \leq -C(\log N)^{1/\alpha}, E_N\right). \end{aligned} \quad (17)$$

Note that we have used (16) in the second inequality. Next,

$$\begin{aligned} &\mathbb{P}\left(\sup_{n=N_0+1,\dots,N} S_n \leq -C(\log N)^{1/\alpha}, E_N\right) \\ &\geq \mathbb{P}\left(\sup_{n=N_0+1,\dots,N} S_n \leq -C(\log N)^{1/\alpha}\right) - \mathbb{P}(E_N^c) \\ &\geq \mathbb{P}\left(\sup_{n=N_0+1,\dots,N} S_n - S_{N_0} \leq 0, S_{N_0} \leq -C(\log N)^{1/\alpha}\right) - \mathbb{P}(E_N^c) \\ &\geq \mathbb{P}\left(\sup_{n=1,\dots,N} S_n \leq 0\right) \mathbb{P}(S_{N_0} \leq -C(\log N)^{1/\alpha}) - \mathbb{P}(E_N^c). \end{aligned}$$

Let $N_0 := \lfloor \log N \rfloor^{2/\alpha}$. Then $\mathbb{P}(S_{N_0} \leq -C(\log N)^{1/\alpha}) \geq \mathbb{P}(S_{N_0}/\sqrt{N_0} \leq -C)$ and the r.h.s. converges to a constant by the CLT. Using the estimate on $\mathbb{P}(E_N^c)$ from above and (17), we have for N large enough that

$$p_N \geq C_1 p_{N_0} \cdot N^{-1/2} = C_1 \mathbb{P}\left(\sup_{n=1,\dots,\lfloor \log N \rfloor^{2/\alpha}} X_n \leq 0\right) N^{-1/2}. \quad (18)$$

Since $c_n \geq 0$ for all n , we can now use the trivial estimate $p_{N_0} \geq \mathbb{P}(Y_1 \leq 0)^{N_0} = e^{-\kappa N_0}$ implying for N large enough that

$$p_N \geq C_1 \exp(-\kappa \lfloor \log N \rfloor^{2/\alpha}) N^{-1/2}.$$

Using this as an a priori estimate for p_{N_0} , we get for large N in view of (18) that

$$\begin{aligned} p_N &\geq C_1^2 \exp(-\kappa \lfloor \log N_0 \rfloor^{2/\alpha}) N_0^{-1/2} N^{-1/2} \\ &= C_1^2 \exp(-\kappa \lfloor \log(\lfloor \log N \rfloor^{1/\alpha}) \rfloor^{2/\alpha}) \lfloor \log N \rfloor^{-1/\alpha} N^{-1/2} \\ &\geq C_2 \exp(-C_3 (\log \log N)^{2/\alpha}) (\log N)^{-1/\alpha} N^{-1/2}. \end{aligned}$$

Using this improved estimate again to obtain a lower bound on p_{N_0} , we deduce from (18) that $p_N \gtrsim (\log N)^{-1/\alpha+o(1)} N^{-1/2}$. \square

Remark 4.3. One cannot expect to get a useful lower bound without any restriction on the weights c_n . For instance, if Y_1 takes only values ± 1 and $X_n = \sum_{k=1}^n c_{n-k} Y_k$ with $c_0 = 1, c_1 = -3$, then $\mathbb{P}(X_1 \leq 0, X_2 \leq 0) = \mathbb{P}(X_1 \leq 0, X_1 + X_2 \leq 0) = 0$.

In order to complete the proof of Theorem 4.2, let us prove the following lemma.

Lemma 4.4. *Let $(f_n)_{n \geq 1}$ denote a sequence of positive numbers with $f_N \rightarrow \infty$ and $f_N/\sqrt{N} \rightarrow 0$ as $N \rightarrow \infty$. Let $(S_n)_{n \geq 1}$ denote a centered random walk with $\mathbb{E}[S_1^2] \in (0, \infty)$ and let $M_n := \max\{S_1, \dots, S_n\}$. There are constants C, N_0 independent of the sequence (f_n) such that*

$$\mathbb{P}(M_N \leq f_N) \leq C f_N N^{-1/2}, \quad f_N, N \geq N_0.$$

Proof. Since independent random variables are associated (Esary et al. (1967)), we have for $1 \leq N_0 < N$ that

$$\mathbb{P}(S_n \leq 0, \forall n = 1, \dots, N) \geq \mathbb{P}(S_n \leq 0, \forall n = 1, \dots, N_0) \mathbb{P}(S_n \leq 0, \forall n = N_0 + 1, \dots, N).$$

Now

$$\begin{aligned} \mathbb{P}\left(\sup_{n=N_0+1, \dots, N} S_n \leq 0\right) &\geq \mathbb{P}\left(S_{N_0} \leq -f_N, \sup_{n=N_0+1, \dots, N} S_n - S_{N_0} \leq f_N\right) \\ &= \mathbb{P}(S_{N_0} \leq -f_N) \mathbb{P}\left(\sup_{n=1, \dots, N-N_0} S_n \leq f_N\right) \geq \mathbb{P}(S_{N_0} \leq -f_N) \mathbb{P}(M_N \leq f_N). \end{aligned}$$

Hence, we get that

$$\mathbb{P}(M_N \leq f_N) \leq \frac{\mathbb{P}(M_N \leq 0)}{\mathbb{P}(M_{N_0} \leq 0) \mathbb{P}(S_{N_0} \leq -f_N)}.$$

With $N_0 = \lfloor f(N) \rfloor^2$, it follows from the CLT that $\mathbb{P}(S_{N_0} \leq -f_N) \rightarrow \mathbb{P}(Z \leq -1)$ where Z is a centered Gaussian with variance $\mathbb{E}[Y_1^2]$. Moreover, since $\mathbb{P}(M_N \leq 0) \sim cN^{-1/2}$, we conclude that

$$\frac{\mathbb{P}(M_N \leq 0)}{\mathbb{P}(M_{N_0} \leq 0) \mathbb{P}(S_{N_0} \leq -f_N)} \sim \frac{N^{-1/2}}{N_0^{-1/2} \mathbb{P}(Z \leq -1)} \sim f_N N^{-1/2} / \mathbb{P}(Z \leq -1).$$

□

Corollary 4.5. *Assume that $\mathbb{E}[Y_1] = 0$. Let $a_1 + a_2 = 1$ with $|a_2| < 1$ and $x \geq 0$.*

1. *If $|Y_1| \leq M$ a.s., it holds that $p_N(x) \asymp N^{-1/2}$ as $N \rightarrow \infty$.*
2. *If $\mathbb{E}[\exp(|Y_1|^\alpha)] < \infty$ for some $\alpha > 0$, it holds that $p_N(x) = N^{-1/2+o(1)}$ as $N \rightarrow \infty$.*

Proof. If X is AR(2) with coefficients a_1, a_2 as in the statement of the corollary, we have seen that $X_n = \sum_{k=1}^n Z_k$ where Z is AR(1) with $Z_n = -a_2 Z_{n-1} + Y_n$, i.e. $Z_n = \sum_{k=1}^n (-a_2)^{n-k} Y_k$. Since $\sum_{k=0}^n (-a_2)^k > 0$ for all n , it is not hard to see that part 2 and part 3 of Theorem 4.2 imply part 2 of the corollary. Similarly, by part 1 of Theorem 4.2 and the fact that $p_N(x) \asymp p_N$ (see the comment at the end of Section 2), we obtain part 1 of the corollary. \square

In analogy to the results for random walks, it is very likely that the assertion of Corollary 4.5 remains true under the much weaker integrability assumption $\mathbb{E}[Y_1^2] \in (0, \infty)$. Depending on the sign of a_1 , we can improve the preceding corollary by proving an upper or lower bound of order $N^{-1/2}$:

Proposition 4.6. *Let $a_1 + a_2 = 1$ with $|a_2| < 1$. Assume that $\mathbb{E}[Y_1] = 0$, $\mathbb{E}[Y_1^2] \in (0, \infty)$.*

1. *If $a_2 > 0$, we have that $p_N(x) \lesssim N^{-1/2}$ for all $x \geq 0$.*
2. *If $a_2 < 0$, we have that $p_N(x) \gtrsim N^{-1/2}$ for all $x \geq 0$.*

Proof. For $n \geq 1$, set $S_n := X_n + a_2 X_{n-1}$ and note that

$$S_n = a_1 X_{n-1} + a_2 X_{n-2} + Y_n + a_2 X_{n-1} = X_{n-1} + a_2 X_{n-2} + Y_n = S_{n-1} + Y_n,$$

i.e. $(S_n)_{n \geq 1}$ defines a centered random walk. Moreover, since $a_1 + a_2 = 1$, we have, for $n \geq 1$, that

$$X_n = (a_1 - 1)X_{n-1} + X_{n-1} + a_2 X_{n-2} + Y_n = (a_1 - 1)X_{n-1} + S_{n-1} + Y_n = -a_2 X_{n-1} + S_n.$$

In particular, if $a_2 > 0$, it holds that $X_n \leq x$ for $n = 1, \dots, N$ implies that $S_n \leq a_2 x$ for $n = 1, \dots, N$ and therefore,

$$p_N(x) \leq \mathbb{P} \left(\sup_{n=1, \dots, N} S_n \leq a_2 x \right) \lesssim N^{-1/2}.$$

Similarly, if $a_2 < 0$, $S_n \leq 0$ for $n = 1, \dots, N$ implies that $X_n \leq 0$ for $n = 1, \dots, N$, which yields the lower bound. \square

Let us finally remark that Theorem 4.2 is also applicable to integrated AR(p)-processes such that the roots s_1, \dots, s_p of the corresponding characteristic polynomial lie inside the unit disc. Let us just state the simplest case of bounded innovations Y_n . Set

$$\Delta_p := \left\{ (a_1, \dots, a_p) : \max_{k=1, \dots, p} |s_k| < 1 \right\}$$

where s_1, \dots, s_p are the roots of the characteristic polynomial, see p. 6.

Corollary 4.7. *Let X be the $AR(p)$ -process corresponding to $(a_1, \dots, a_p) \in \Delta_p$. Assume that $|Y_1| \leq M < \infty$ a.s. Then there is $x_0 \geq 0$ such that for all $x \geq x_0$, we have that*

$$\mathbb{P} \left(\sup_{n=1, \dots, N} \sum_{k=1}^n X_k \leq x \right) \asymp N^{-1/2}.$$

Since we know the region Δ_2 explicitly (cf. Figure 2.1), we obtain the following result for $AR(3)$ -processes:

Corollary 4.8. *Let X be $AR(3)$ with a_1, a_2, a_3 satisfying*

$$a_1 + a_2 + a_3 = 1, \quad a_2 < \min \{1, 3 - 2a_1\}, \quad a_2 > -a_1.$$

Assume that $|Y_1| \leq M$ a.s. for some $M < \infty$. Then there is $x_0 \geq 0$ such that $p_N(x) \asymp N^{-1/2}$ for all $x \geq x_0$.

Proof. Let us show that X is an integrated $AR(2)$ -process \tilde{X} with parameters in Δ_2 . Since $a_1 + a_2 + a_3 = 1$, we have that $T_2(a_1 - 1, a_1 + a_2 - 1) = (a_1, a_2, a_3)$ where T_2 was defined in (15). Hence, by Corollary 4.7, we only need to show that

$$(a_1 - 1, a_1 + a_2 - 1) \in \Delta_2 = \{(\tilde{a}_1, \tilde{a}_2) : \tilde{a}_1 + \tilde{a}_2 < 1, \tilde{a}_2 < 1 + \tilde{a}_2, \tilde{a}_2 > -1\},$$

(see Remark 2.1) whenever (a_1, a_2, a_3) satisfy the constraints stated in the corollary. Let $\tilde{a}_1 = a_1 - 1$ and $\tilde{a}_2 = a_1 + a_2 - 1$. Now $a_2 < 3 - 2a_1$ amounts to $\tilde{a}_1 + \tilde{a}_2 = 2a_1 + a_2 - 2 < 1$. Next, $\tilde{a}_2 < 1 + \tilde{a}_1$ is equivalent to $a_2 < 1$, whereas $\tilde{a}_2 > -1$ translates into $a_1 > -a_2$. \square

4.2 The case $a_1 = 0$

We still have to consider the case $X_n = X_{n-2} + Y_n$ which is a special case of the equation $X_n = \rho X_{n-2} + Y_n$. The solution of the latter equation is given by

$$X_n = \begin{cases} \sum_{j=1}^k \rho^{k-j} Y_{2j-1}, & n = 2k - 1, k \in \mathbb{N}, \\ \sum_{j=1}^k \rho^{k-j} Y_{2j}, & n = 2k, k \in \mathbb{N}. \end{cases}$$

In particular, (X_{2n}) and (X_{2n-1}) define two independent sequences with the same law as $(Z_n)_{n \geq 1}$ given by $Z_n = \rho Z_{n-1} + Y_n$. Hence,

$$\begin{aligned} \mathbb{P}(\sup_{n=1, \dots, 2N} X_n \leq x) &= (\mathbb{P}(\sup_{n=1, \dots, N} Z_n \leq x))^2, \\ \mathbb{P}(\sup_{n=1, \dots, 2N-1} X_n \leq x) &= \mathbb{P}(\sup_{n=1, \dots, N} Z_n \leq x) \mathbb{P}(\sup_{n=1, \dots, N-1} Z_n \leq x). \end{aligned} \tag{19}$$

In particular, the behaviour of the survival probability can be determined by the survival probabilities of $AR(1)$ -processes. If $\rho = 1$, X defines two independent random walk, so we immediately obtain the following lemma:

Lemma 4.9. *Let $\mathbb{E}[Y_1] = 0$, $\mathbb{E}[Y_1^2] \in (0, \infty)$. $X_n = X_{n-2} + Y_n$. Then for any $x \geq 0$, there is a constant $c(x)$ such that*

$$\mathbb{P}\left(\sup_{n=1,\dots,N} X_n \leq x\right) \sim c(x)N^{-1}, \quad N \rightarrow \infty.$$

Proof. By the preceding discussion, (X_{2n}) and (X_{2n-1}) define two independent centered random walks with finite variance that have the same law. It is then well known that $\mathbb{P}\left(\sup_{n=1,\dots,N} \sum_{k=1}^n Y_k \leq x\right) \sim d(x)N^{-1/2}$. The assertion follows in view of (19). \square

Remark 4.10. By the same reasoning, if $X_n = X_{n-p} + Y_n$ ($p \geq 1$), we have that $p_N(x) \sim c(x)N^{-p/2}$ for any $x \geq 0$ if $\mathbb{E}[Y_1] = 0$ and $\mathbb{E}[Y_1^2] \in (0, \infty)$.

5 A positive limit

We now turn to the case that the survival probability converges to a positive limit, i.e. $p_N(x) \rightarrow p_\infty(x) > 0$ as $N \rightarrow \infty$, implying that the process $(X_n)_{n \geq 1}$ stays below x at all times with positive probability. If $X_n = \sum_{k=1}^n c_{n-k}Y_k$, one would expect that this happens if $0 < c_n \rightarrow \infty$ and $c_n - c_{n-1} \rightarrow \infty$. Indeed, if c_n is very large compared to c_k for $k \leq n-1$, then $Y_1 \leq -\delta$ for some $\delta > 0$ implies that $X_n \leq -\delta c_n + \sum_{k=2}^n c_{n-k}Y_k$, and one expects that the expression on the r.h.s. stays below a fixed barrier with high probability. In fact, we can transform this idea directly into a proof.

Proposition 5.1. *Let $(\alpha_n)_{n \geq 0}$ denote a sequence of positive numbers. Let $\rho > 1$ and assume that $\mathbb{P}(Y_1 < 0) > 0$ and $\mathbb{P}(Y_1 \geq x) \lesssim (\log x)^{-\alpha}$ as $x \rightarrow \infty$ for some $\alpha > 1$. Let $X_n := \sum_{k=1}^n \rho^{n-k} \alpha_{n-k} Y_k$.*

1. *If $(\alpha_n)_{n \geq 0}$ is nondecreasing, there is a constant $c > 0$ such that*

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} \{X_n \leq -c\alpha_{n-1}\rho^{n-1}\}\right) > 0.$$

2. *If $0 < l \leq \alpha_n \leq u < \infty$ for all $n \geq 0$, there is a constant $c > 0$ such that*

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} \{X_n \leq -c\rho^{n-1}\}\right) > 0.$$

Proof. We first prove part 1. Let $\delta > 0$ such that $\mathbb{P}(Y_1 \leq -\delta) > 0$. Let $\beta > 0$ denote a sequence of positive numbers with $\beta \sum_{k=1}^{\infty} k^{-2} \leq \delta/2$. Then

$$A_N := \{Y_1 \leq -\delta\} \cap \bigcap_{n=2}^N \{Y_n \leq \rho^{n-1}\beta n^{-2}\} \subseteq \bigcap_{n=1}^N \{X_n \leq -\delta\alpha_{n-1}\rho^{n-1}/2\}$$

Indeed, since (α_n) is nondecreasing, the event A_N implies that $X_1 = \alpha_0 Y_1 \leq -\alpha_0 \delta$ and for all $n = 2, \dots, N$ that

$$\begin{aligned} X_n &= \rho^{n-1} \alpha_{n-1} Y_1 + \sum_{k=2}^n \rho^{n-k} \alpha_{n-k} Y_k \leq -\delta \alpha_{n-1} \rho^{n-1} + \rho^{n-1} \sum_{k=2}^n \alpha_{n-k} \beta k^{-2} \\ &\leq -\delta \alpha_{n-1} \rho^{n-1} + \rho^{n-1} \alpha_{n-1} \beta \sum_{k=1}^{\infty} k^{-2} = \alpha_{n-1} \rho^{n-1} \left(\beta \sum_{k=1}^{\infty} k^{-2} - \delta \right) \leq -\delta \alpha_{n-1} \rho^{n-1} / 2. \end{aligned}$$

Finally, in view of the assumption on the tail behaviour of Y_1 , it is not hard to show that

$$\lim_{N \rightarrow \infty} \mathbb{P}(A_N) = \mathbb{P}(Y_1 \leq -\delta) \lim_{N \rightarrow \infty} \prod_{n=2}^N (1 - \mathbb{P}(Y_1 > \beta \rho^{n-1} n^{-2})) > 0.$$

The proof of part 2 is very similar. Let A_N be defined as above. Then, using the bounds on (α_n) , we get for $n = 2, \dots, N$ that

$$\begin{aligned} X_n &\leq -\delta \alpha_{n-1} \rho^{n-1} + \rho^{n-1} \sum_{k=2}^n \alpha_{n-k} \beta k^{-2} \leq -\delta l \rho^{n-1} + \rho^{n-1} u \beta \sum_{k=1}^{\infty} k^{-2} \\ &= \rho^{n-1} \left(\beta u \sum_{k=1}^{\infty} k^{-2} - \delta l \right). \end{aligned}$$

For $\beta > 0$ sufficiently small, this implies that $X_n \leq -(\delta l/2) \rho^{n-1}$ for all $n = 2, \dots, N$. \square

We can now prove Theorem 1.3 showing that the survival probability converges to a positive constant if X is AR(2) with $(a_1, a_2) \in C$ (cf. Figure 1.1) under mild conditions.

Proof. (of Theorem 1.3) Let $(a_1, a_2) \in C$. Assume first that $a_1 > 0$ and $a_2 \in \mathbb{R}$ such that $a_1^2 + 4a_2 > 0$. Moreover, assume that either $a_1 \geq 2$ or $a_1 + a_2 > 1$ if $a_1 < 2$. Recall from (2) that $c_n = s_1^{n+1}/h - s_2^{n+1}/h$ where $h > 0$ since $a_1^2 + 4a_2 > 0$. Note that $s_1 = (a_1 + h)/2 > 1$ if and only if either $a_1 \geq 2$ or if $a_1 + a_2 > 1$ in case $a_1 < 2$. Moreover $|s_2| < s_1$ if and only if $a_1 > 0$ and $h > 0$. Hence, in view of our assumptions, we have that $c_n = s_1^n s_1 / h (1 - (s_2/s_1)^{n+1}) =: s_1^n \alpha_n \geq 0$ for all n . Note that $\alpha_n \rightarrow s_1/h > 0$. Hence, the assertion follows by part 2 of Proposition 5.1.

If $a_1^2 + 4a_2 = 0$ and $a_1 > 2$, $c_n = (a_1/2)^n (n+1)$ by (2). Hence, the result follows from part 1 of Proposition 5.1 with $\rho = a_1/2 > 1$ and $\alpha_n = n+1$.

Finally, if $a_1 = 0$ and $a_2 > 1$, the claim follows in view of (19) and Proposition 5.1. \square

References

F. Aurzada and C. Baumgarten. Survival probabilities of weighted random walks. *ALEA, Lat. Am. J. Probab. Math. Stat.*, 8:235–258, 2011.

- F. Aurzada and T. Simon. Persistence probabilities and exponents. *Preprint*, 2012.
- P. J. Brockwell and R. A. Davis. *Time series: theory and methods*. Springer Series in Statistics. Springer-Verlag, New York, 1987.
- A. Dembo, B. Poonen, Q.-M. Shao, and O. Zeitouni. Random polynomials having few or no real zeros. *J. Amer. Math. Soc.*, 15(4):857–892 (electronic), 2002.
- A. Dembo, J. Ding, and F. Gao. Persistence of iterated partial sums. *Preprint*, 2012.
- R. A. Doney. On the asymptotic behaviour of first passage times for transient random walk. *Probab. Theory Related Fields*, 81(2):239–246, 1989.
- S. N. Elaydi. *An introduction to difference equations*. Undergraduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1999.
- J. D. Esary, F. Proschan, and D. W. Walkup. Association of random variables, with applications. *Ann. Math. Statist.*, 38:1466–1474, 1967.
- W. Feller. *An introduction to probability theory and its applications. Vol. II*. Second edition. John Wiley & Sons Inc., New York, 1971.
- G. Grimmett and D. Stirzaker. *One thousand exercises in probability*. Oxford University Press, Oxford, 2001.
- M. Ledoux and M. Talagrand. *Probability in Banach spaces*. Springer-Verlag, Berlin Heidelberg New York, 1991.
- W. Li and Q. Shao. Lower tail probabilities for Gaussian processes. *Ann. Probab.*, 32(1A):216–242, 2004.
- W. V. Li and Q.-M. Shao. Recent developments on lower tail probabilities for Gaussian processes. *Cosmos*, 1(1):95–106, 2005.
- E. Lukacs. *Characteristic functions*. Hafner Publishing Co., New York, 1970. Second edition, revised and enlarged.
- A. Novikov and N. Kordzakhia. Martingales and first passage times of AR(1) sequences. *Stochastics*, 80(2-3):197–210, 2008.
- Y. Peres, W. Schlag, and B. Solomyak. Sixty years of Bernoulli convolutions. In *Fractal geometry and stochastics, II (Greifswald/Koserow, 1998)*, volume 46 of *Progr. Probab.*, pages 39–65. Birkhäuser, Basel, 2000.
- Y. G. Sinaĭ. Statistics of shocks in solutions of inviscid Burgers equation. *Comm. Math. Phys.*, 148(3):601–621, 1992.